Functional Analysis, PS9

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(1) $A-\lambda I$ is not injective $\Longleftrightarrow$

$A-\lambda I$ not injective OR $A-\lambda I$ not surjective

i.e. $\exists x \neq 0$

$Ax = \lambda x$

$\lambda \in \sigma(A)$
[point spectrum, eigenvalue)

i.e.

$R(A-\lambda I) \neq H$

at least $R(A-\lambda I)$ is dense in H

[CONTINUOUS]

$R(A-\lambda I)$ is a closed subspace of H.

(2) $A : \mathbb{C}^n \to \mathbb{C}^n$. Then $A-\lambda I$ is a map between finite dimensional spaces and so $(A-\lambda I)$ is injective iff $(A-\lambda I)$ is surjective.

$$\dim \ker(A-\lambda I) + \dim R(A-\lambda I) = \dim \mathbb{C}^n$$

$\Rightarrow$ A has purely point spectrum.

3 $T - \lambda I = -\lambda \left(I - \frac{T}{\lambda} \right)$

if $|\lambda| > \|T\|$ then this operator is invertible

$\Rightarrow |\sigma(T)| \leq \|T\| \Rightarrow \sigma(T)$ is bounded

From lecture $\rho(T)$ is open $\Rightarrow \sigma(T)$ is closed

$\Rightarrow \sigma(T)$ is compact.

4 $\exists \{x_n\} \; \exists_{\varepsilon_n \to 0}$ s.t. $\|Ax_n\| \leq \varepsilon_n \|x_n\|$

$\Rightarrow$ A is NOT invertible.

inverse mapping thm

Suppose A is invertible $\Rightarrow A^{-1}$ is bounded $\Rightarrow$

$\Rightarrow \|y_n\| \leq \varepsilon_n \|A^{-1} y_n\| \Rightarrow \frac{1}{\varepsilon_n} \leq \|A^{-1}\|$

contradiction

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$$M : L^2(0,1) \to L^2(0,1) \quad \text{over } \mathbb{C}.$$

$$Mf(x) = x f(x)$$

Two strategies:

(1) point spectrum (eigenvalues)

$$\exists_{\lambda} \ \exists_{f \neq 0} \ (M - \lambda I) f = 0 \qquad Mf = \lambda f$$

$$x f(x) = \lambda f(x) \quad \Rightarrow (x - \lambda) f(x) = 0 \ \Rightarrow f = 0$$

$\Rightarrow M$ has no eigenvalues (empty point sp.)

(2) try to invert $M - \lambda I$

$$(M - \lambda I) f(x) = x f(x) - \lambda f(x) = (x - \lambda) f(x)$$

$$(M - \lambda I)^{-1} f = \frac{1}{x - \lambda} f(x) \quad \text{if } \lambda \notin [0,1]$$

It is bdd $\|(M - \lambda I)^{-1} f\|_2^2 = \int \frac{1}{|x-\lambda|^2} |f(x)|^2 \, dx$

$$\exists_c \ \inf |x - \lambda|^2 \geq c > 0 \qquad \cdots \leq \frac{1}{c^2} \|f\|_2^2 .$$

Hence, $\sigma(M) \subset [0,1]$.

To prove $\sigma(M) = [0,1]$ we prove that M is not surjective i.e. $R(M) \neq L^2(0,1)$.

Suppose $\exists_{f \in L^2(0,1)}$ $1 = (M - \lambda I) f \Rightarrow$

$$1 = (x-\lambda) f(x) \Rightarrow f(x) = \frac{1}{x-\lambda} \notin L^2(0,1).$$

for $\lambda \in [0,1]$.

(In fact, $\sigma(M)$ is purely continuous. Indeed, we need $R(M-\lambda I)$ is dense in H.

Let $f \in H$, $f_n(x) := f(x) 1\!\!1_{|x-\lambda| \geq \frac{1}{n}}$. Then f_n is in $R(M-\lambda I)$ ($M g_n = f_n$ for $g_n = \frac{f_n}{(x-\lambda)}$).

By dominated convergence, $f_n \to f$ in $L^2(0,1)$.

$\square$.

6 $A: \ell^2 \to \ell^2$

$$A(x_1, x_2, \dots) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right)$$

(1) point $(A-\lambda I)x = 0 \iff Ax = \lambda x$

$$\left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right) = \lambda(x_1, x_2, x_3, x_4, \dots)$$

If $\lambda = 0 \Rightarrow x = 0$.

Otherwise $x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow \dots \Rightarrow x = 0$.

(2) $(A-\lambda I)x = \left(-\lambda x_1, x_1 - \lambda x_2, \frac{x_2}{2} - \lambda x_3, \frac{x_3}{3} - \lambda x_4, \dots\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

Fix $y \in \ell^2$. Find $x \in \ell^2$ s.t. $\qquad (y_1, y_2, \dots)$

If $\lambda \neq 0$ we can find $y_1, y_2, \dots$ and we need to check they form an element of ℓ^2.

$$x_1 = \frac{-y_1}{\lambda} \qquad \frac{x_k}{k} - \lambda x_{k+1} = y_k$$

$$x_1 = -\frac{y_1}{\lambda} \qquad x_{k+1} = -\frac{y_k}{\lambda} - \frac{x_k}{k\lambda}$$

For $n, m \geq N$

$$\sum_{k=n}^{m} |x_{k+1}|^2 \leq 2\left[\frac{1}{\lambda^2}\sum_{k=n}^{m}|y_k|^2 + \frac{1}{N^2\lambda^2}\sum_{k=n}^{m}|x_k|^2\right]$$

$$\underbrace{\sum_{k=n+1}^{m+1} |x_k|^2} \leq 2\left[\frac{1}{\lambda^2}\sum_{k=n}^{m}|y_k|^2 + \frac{1}{N^2\lambda^2}\underbrace{\sum_{k=n}^{m}|x_k|^2}_{\leq \sum_{k=n}^{m+1}|x_k|^2}\right]$$

$$\Rightarrow -\frac{1}{N^2\lambda^2}|x_n|^2 + \left(\sum_{k=n+1}^{m+1}|x_k|^2\right)\left(1-\frac{2}{N^2\lambda^2}\right)$$

$$\leq \frac{2}{\lambda^2}\sum_{k=n}^{m}|y_k|^2$$

Choose N so that $1-\frac{2}{N^2\lambda^2} \geq \frac{1}{2}$. Send $m \to \infty$ $\Rightarrow$ $x \in \ell^2$. $\Rightarrow$ $\sigma(A) \subset \{0\}$.

Finally, we need to study $\lambda = 0$. Clearly $R(A)$ is a closed subspace of ℓ^2. $\square$.

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$(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$

$(\dots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, \dots$

• (eigenvalues)

$$Rx = \lambda x \quad \underset{\substack{x \in \ell^2 \\ x \neq 0}}{\exists} \quad \underset{\lambda \in \mathbb{C}}{\partial}.$$

$$\Rightarrow (Rx)_k = x_{k-1} = \lambda x_k \Rightarrow x_k = \frac{x_{k-1}}{\lambda}$$

$$x = \left(\dots, \lambda^2 x_0, \lambda x_0, x_0, \frac{x_0}{\lambda}, \frac{x_0}{\lambda^2}, \dots \right)$$

$\notin \ell^2$ unless $\lambda = 0$.

$\Rightarrow$ no eigenvalues.

• (try to invert)

$$R - \lambda I = -\lambda \left(I - \frac{R}{\lambda} \right) \text{ invertible when}$$

$$\left\| \frac{R}{\lambda} \right\| < 1 \Leftrightarrow \|R\| < |\lambda| \Leftrightarrow 1 < |\lambda|.$$

So $R - \lambda I$ is invertible for $1 < |\lambda|$.

Now: $R-\lambda I = R(I-\lambda L)$ invertible when $\|\lambda L\| < 1 \iff |\lambda| < 1$.

It follows that $\sigma(R) \subset \{|\lambda| = 1\}$.

Proof that $\sigma(R) = \{|\lambda| = 1\}$. Fix λ, $|\lambda| = 1$.

$$x_n = (\dots, 0, \bar{\lambda}, \bar{\lambda}^2, \dots, \bar{\lambda}^n, 0, \dots)$$

$$\lambda x_n = (\dots, 0, 1, \bar{\lambda}, \dots, \bar{\lambda}^{n-1}, 0, \dots)$$

$$R x_n = (\dots, 0, 0, \bar{\lambda}, \dots, \bar{\lambda}^{n-1}, \bar{\lambda}^n, 0, \dots)$$

$$(R-\lambda I) x_n = (\dots, 0, -1, 0, \dots, 0, \bar{\lambda}^n, 0, \dots)$$

$$\|x_n\|_{\ell^2} = n \qquad \|(R-\lambda I)x_n\|_{\ell^2} = 2.$$

$$\|(R-\lambda I)x_n\|_{\ell^2} = 2 = \frac{2}{n}\cdot n = \underbrace{\frac{2}{n}}_{\to 0}\|x_n\|_{\ell^2}.$$

$\Rightarrow \sigma(R) = \{|\lambda| = 1\}$.