Functional Analysis (WS 2020/21) Special Problems Solutions

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1 Invitation to Sobolev Spaces

One of the most fundamental topic in analysis is the notion of Sobolev spaces and weak derivatives. Let $f \in L^p(0,1)$ where $1 \leqslant p \leqslant \infty$. We say that f is weakly differentiable if there is a function $g \in L^p(0,1)$ such that

See COMMENT 3.

$$\int_0^1 f(x)\Phi'(x)\mathrm{d}x = -\int_0^1 g(x)\Phi(x)\mathrm{d}x$$

for all $\Phi \in C_c^\infty(0,1)$ (i.e. smooth functions $\Phi : [0,1] \to \mathbb{R}$ with compact support in (0,1)). If this is the case, we write $f' = g$ and we say that g is the weak (Sobolev) derivative of f. The space of all weakly differentiable functions in this sense is denoted with $W^{1,p}(0,1)$ and is called Sobolev space. In what follows, we gonna check that weak derivatives make sense (they coincide with strong derivatives whenever the latter exist) and they don't see what happens on small sets (i.e. sets of measure zero). Finally, functional analytic properties of Sobolev spaces will be established.

(A) Uniqueness

Suppose that h and g are weak derivatives of $f \in L^p(0,1)$. For all $\Phi \in C_c^\infty(0,1)$ we have

$$\int_0^1 f(x)\Phi'(x)\mathrm{d}x = -\int_0^1 g(x)\Phi(x)\mathrm{d}x = -\int_0^1 h(x)\Phi(x)\mathrm{d}x \tag{1}$$

Then, we can simply deduct that

$$\int_0^1 [g(x) - h(x)]\,\Phi(x)\mathrm{d}x = 0 \tag{2}$$

And therefore

$$g(x) - h(x) = 0 \quad \text{a.e.} \tag{3}$$

See COMMENT 1 and 2 on how to get (3) from (2).

(B) Coinciding with strong derivative

Suppose that $f \in L^p(0,1) \cap C^1[0,1]$. Let's denote it's strong derivative by F. Let $\Phi \in C_c^\infty(0,1)$. Using the integration by parts:

$$\int_0^1 f(x)\Phi'(x)\mathrm{d}x = f\Phi\Big|_0^1 - \int_0^1 F(x)\Phi(x)\mathrm{d}x \tag{4}$$

We know that $\Phi : [0,1] \to \mathbb{R}$ has compact support $C \subseteq (0,1)$; $\Phi(0) = \Phi(1) = 0$ and we conclude that

$$\int_0^1 f(x)\Phi'(x)\mathrm{d}x = -\int_0^1 F(x)\Phi(x)\mathrm{d}x \tag{5}$$

which means that F is also weak derivative of f.

(C) Example of $f \in W^{1,p}(0,1) \setminus C^1(0,1)$

Let's take

$$f = \left|x - \frac{1}{2}\right|, \qquad F = \begin{cases} -1, & x < \frac{1}{2} \\ 0, & x = \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases} \tag{6}$$

It's not in $C^1(0,1)$ obviously, but for all $x \neq \frac{1}{2}$ we have $f'(x) = F$ in terms of strong derivative. The fact that it is weak derivative follows from

$$\int_0^1 f\Phi' = \int_0^{\frac{1}{2}} f\Phi' + \int_{\frac{1}{2}}^1 f\Phi' = -\int_0^{\frac{1}{2}} F\Phi - \int_{\frac{1}{2}}^1 F\Phi \tag{7}$$

The latter two obviously exist (F is constant in terms of these integrals) and thereby it is equal to $-\int_0^1 F\Phi$, namely

$$\int_0^1 f\Phi' = -\int_0^1 F\Phi \tag{8}$$

(D) Trivial derivative implies that the function is constant

Suppose that $f' = 0$ a.e. We have

$$\int_0^1 f\Phi' = -\int_0^1 f'\Phi = 0 \tag{9}$$

for all $\Phi \in C_c^\infty(0,1)$ Let's fix Φ and denote

$$\Phi = A\theta + \xi' \tag{10}$$

where $A \in \mathbb{R}$ and $\theta, \psi \in C_c^\infty(0,1)$ fulfill following conditions:

- $\int_0^1 \theta = 1$

- $A = \int_0^1 \Phi$
- $\xi(x) = \int_0^x \Phi(t) - A\theta(t)\mathrm{d}t$

We have

$$\int_0^1 f\Phi = A\int_0^1 f\theta + f\psi' = c\int_0^1 \Phi \quad \text{where} \quad c = \int_0^1 f\theta \tag{11}$$

Therefore

$$\int_0^1 f\Phi = c\int_0^1 \Phi \Rightarrow f = c \tag{12}$$

where all equalities are taken a.e.

(E) Norm and Completeness

By definition of $W^{1,p}(0,1)$ we know that

$$\|f\|_{1,p} = \|f\|_p + \|f'\|_p \tag{13}$$

is properly defined norm (both $f, f' \in L^p(0,1)$). Let us now show it's Banach space. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ - Cauchy sequence in $W^{1,p}(0,1)$. Therefore we know that both $\{f_n\}_{n\in\mathbb{N}}$ and $\{f_n'\}_{n\in\mathbb{N}}$ must be Cauchy sequences in $L^p(0,1)$. Let's denote their limits by F, g respectively. We will show that $f_n \xrightarrow{\|\cdot\|_{1,p}} F$.

$$\|F - f_n\|_{1,p} = \|F - f_n\|_p + \|g - f_n\|_p \tag{14}$$

Taking the limit as n goes to infinity we have

$$\|F - f_n\|_{1,p} \xrightarrow{n\to\infty} 0 + 0 = 0 \tag{15}$$

So we indeed have convergence. Let $\Phi \in C_c^\infty(0,1)$. We need

$$\int_0^1 F\Phi' = -\int_0^1 g\Phi \tag{16}$$

We will argue, that

$$-\int_0^1 g\Phi = -\lim_{n\to\infty}\int_0^1 f_n'\Phi = \lim_{n\to\infty}\int_0^1 f_n\Phi' = \int_0^1 F\Phi' \tag{17}$$

It suffices to say that relation $g \mapsto \int_0^1 g\Phi$ describes bounded operator. It is because

$$\sup_{\|g\|_p=1}\left|\int_0^1 g\Phi\right| \leqslant \sup_{\|g\|_p=1}\int_0^1 |g|\,|\Phi| \leqslant \sup_{\|g\|_p=1}\int_0^1 |g\Phi| < \infty \tag{18}$$

Where the latter inequality follows from the fact that $g\Phi \in L^p \subseteq L^1$

(17) follows also from Hölder[3] inequality, for example

$\int_0^1 (f_n' - g)\phi \leq \|f_n' - g\|_p \cdot \|\phi\|_{p'} \to 0$ from L^p convergence.

COMMENT 1. Elementary sln to (A)

To see that (2) implies (3), we can observe that (2) implies $\int_A g-h = 0$ for all subintervals $A \subset (0,1)$ $\rightsquigarrow$ standard approximation and dominated convergence.

Now, suppose that there is $E \subset [\alpha, \beta] \subset\subset [0,1]$ s.t. $g-h>0$ on E and $\lambda(E)>0$. By inner regularity of Lebesgue measure, there is closed $F \subset E$ s.t. $\lambda(F)>0$. It follows that $[\alpha,\beta] \setminus F$ is open and so $[\alpha,\beta] \setminus F = \bigcup (a_i, b_i)$ = sum of open intervals. We have

$$0 = \int_\alpha^\beta g-h = \int_F (g-h) + \sum_i \int_{a_i}^{b_i} (g-h) = \int_F (g-h)$$

$\Rightarrow$ contradiction. Similarly, when $g-h<0$.

COMMENT 2. Standard solution to (B).

Let $f = g-h$. Fix δ and consider $\operatorname{sgn}(f) \cdot \mathbb{1}_{[\delta, 1-\delta]}$. Use standard mollifier $\{\eta_\varepsilon\}_\varepsilon$ where $\varepsilon < \delta/2$ to get smooth function

$$\phi_\varepsilon = \left(\operatorname{sgn} f \cdot \mathbb{1}_{[\delta, 1-\delta]}\right) * \eta_\varepsilon \in C_c^\infty(0,1)$$

Note that $\phi_\varepsilon \to \operatorname{sgn} f \, \mathbb{1}_{[\delta, 1-\delta]}$ a.e. when $\varepsilon \to 0$. Hence,

$\int f \phi_\varepsilon = 0$ implies $\int |f| \, \mathbb{1}_{[\delta, 1-\delta]} = 0$. As δ is arbitrary,

$\int_0^1 |f| = 0$ and this implies $f = 0$ a.e. in $[0,1]$.

COMMENT 3.

We can also define $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ where the weak derivative of $f \in W^{1,p}(\Omega)$ is $g = (g_1, \dots, g_n)$

$$\forall_{\varphi \in C_c^\infty(\Omega)} \qquad \int f \cdot \varphi_{x_i} = \int g_i \cdot \varphi$$

so that $g = f_{x_i}$.