On dual spaces of $L^\infty(0,1)$, ℓ^∞ ...

The following steps show that there is no bounded isomorphism between duals of $L^\infty(0,1)$, ℓ^∞ and $L^1(0,1)$, ℓ^1.

STEP 1. Both $L^\infty(0,1)$ and ℓ^∞ contain uncountable subsets $\{x_\alpha\}_{\alpha\in A}$ s.t. $\|x_{\alpha_1} - x_{\alpha_2}\| \geq 2$. Indeed, in ℓ^∞ one can take all binary sequences (we put 0 or 1 on each coordinate).

In $L^\infty(0,1)$ we can perform a similar construction. Consider sequence of intervals $[1/2, 1]$, $[1/4, 1/2]$, $[1/8, 1/4]$, ... and take all functions being 0 or 1 on these intervals. This proves that ℓ^∞, $L^\infty(0,1)$ are not separable.

STEP 2. On the other hand, ℓ^1 and $L^1(0,1)$ are separable. Indeed, for ℓ^1 take Schauder basis $\{e_i\}_{i\in I}$ and consider all finite linear combinations with rational coefficients. Similarly, for $L^1(0,1)$

use density of $C[0,1]$ in $L^1[0,1]$ and then, density of polynomials in $C[0,1]$. Finally consider polynomials with rational coefficients.

STEP 3. Here is a general fact: for a Banach space E we have

$$E^* \text{ is separable } \Rightarrow E \text{ is separable}$$

(the opposite is not true as Steps 1 and 2 shows). For a short and simple proof see the book of Brezis – Theorem 3.26.

STEP 4. If $(\ell^\infty)^* \simeq \ell^1$, $(\ell^\infty)^*$ would be separable. It follows that ℓ^∞ would be separable raising contradiction. Similarly for $L^\infty(0,1)$.

$\square$.