Functional Analysis (WS 20/21), Problem Set 1

(normed and Banach spaces, examples (L^p, C^k, sequences))

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normed spaces, Banach spaes

A1. Example 1. Is $C[0,1]$ (space of continuous functions on $[0,1]$) with supremum norm a normed space? What about $C(\mathbb{R})$? What about $C(0,1)$ (i.e. space of continuous functions on $(0,1)$)?

A2. Example 2. Let $|f|_{C^1} := \|f'\|_\infty$. Is $(C^1[0,1], |\cdot|_{C^1})$ (space of continuously differentiable functions on $[0,1]$) a normed space?

A3. Let $(X, \|\cdot\|_X)$ be a normed space and suppose that $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence.

- Prove that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded: $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.
- Assume that $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $\{x_{n_k}\}_{n_k\in\mathbb{N}}$ such that $x_{n_k} \to x$ in X. Prove that $x_n \to x$ in X.

A4. Let $(X, \|\cdot\|_X)$ be a Banach space and $Y \subset X$ a subset of X. Prove that $(Y, \|\cdot\|_X)$ is a Banach space if and only if Y is closed in X.

A5. Prove that a normed space $(X, \|\cdot\|_X)$ is a Banach space if and only if every absolutely convergent series (i.e. $\sum_{k=1}^{\infty} \|x_k\|_X < \infty$) is convergent in X (i.e. sequence of partial sums is convergent in X).

L^p spaces

We write L^p for $L^p(X, \mathcal{F}, \mu)$ where $\mathcal{F}$ is a σ-algebra and μ is a σ-finite measure on $(X, \mathcal{F})$. We write $\|\cdot\|_p$ for L^p norm. We know that $(L^p, \|\cdot\|_p)$ is a Banach space (for $1 \le p \le \infty$).

B1. Suppose $\mu(X) < \infty$. Check that $L^p \subset L^q$ whenever $q \le p$ and

$$\|f\|_q \le \mu(X)^{\frac{1}{q}-\frac{1}{p}} \|f\|_p.$$

Prove that (in general) assumption $\mu(X) < \infty$ is necessary.

B2. Consider linear space $L^2(0,1)$ equipped with $\|\cdot\|_1$ norm. Is $(L^2(0,1), \|\cdot\|_1)$ a normed space? Is it a Banach space?

B3. (**Littlewood's interpolation inequality**) Let $f \in L^p \cap L^q$ for some $1 \le p, q \le \infty$. Prove that for $r \in [p,q]$ we have $f \in L^r$ and $\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha}$. for some $\alpha \in [0,1]$. *Hint:* let $\alpha \in [0,1]$ and write $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

B4. Let $f \in L^p$. Prove that for any p_0 and p_1 such that $1 \le p_0 < p$ and $p < p_1 < \infty$ there are $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$ such that $f = f_0 + f_1$. Thus, f can be always decomposed for "better" and "worse" part. *Hint:* truncate f at your favourite level.

Spaces of continuous and differentiable functions (C, C^1, C^k, C_0, ...)

C1. Prove that space $C^1[0,1]$ of functions continuously differentiable on $[0,1]$, equipped with the norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$ is a Banach space.

C2. Let $\mathcal{P}$ be the space of all polynomials on $[0,1]$ equipped with supremum norm. Prove that $\mathcal{P}$ is a normed space but it is not a Banach space. Is $\mathcal{P}$ closed in $C[0,1]$? *Hint:* Use Problem A5.

C3. Let $\|f\|_A := \sup_{x\in[0,1]} |f(x)|$. Is $(C^1[0,1], \|\cdot\|_A)$ a normed space? Is it a Banach spaces?

C4. Let $\|f\|_B := |f(0)| + \sup_{x\in[0,1]} |f'(x)|$. Prove that $(C^1[0,1], \|\cdot\|_B)$ is a normed space. Is it a Banach spaces?

C5. Prove that the space $C_0(\mathbb{R})$ of continuous functions "vanishing at infinity" (i.e. $f(x) \to 0$ whenever $|x| \to \infty$) equipped with the supremum norm is a Banach space.

C6. Consider space $C_{LIP}[0,1]$ of Lipschitz continuous functions on $[0,1]$, i.e. of functions $f \in C[0,1]$ such that

$$|f|_{LIP} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

a. Is $(C_{LIP}[0,1], |\cdot|_{LIP})$ a normed space? Is it a Banach space?

b. Is $(C_{LIP}[0,1], \|\cdot\|_\infty)$ a normed space? Is it a Banach space?

c. Is $(C_{LIP}[0,1], \|\cdot\|_\infty + |\cdot|_{LIP})$ a normed space? Is it a Banach space?

C7. For $\alpha \in (0,1)$ we define space $C^\alpha[0,1]$ of Hölder continuous functions with exponent α, i.e. of functions $f \in C[0,1]$ such that

$$|f|_\alpha := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

In this problem we take (for simplicity) $\alpha = \frac{1}{2}$. Are the following pairs normed spaces? Are they Banach spaces? Justify your answer.

a. $(C^{1/2}[0,1],\ |\cdot|_{LIP})$

b. $(C^{1/2}[0,1],\ |\cdot|_{1/2})$

c. $(C^{1/2}[0,1],\ \|\cdot\|_\infty + |\cdot|_{LIP})$

d. $(C^{1/2}[0,1],\ \|\cdot\|_\infty + |\cdot|_{1/2})$

e. $(C^{1/2}[0,1],\ \|\cdot\|_\infty + |\cdot|_{LIP} + |\cdot|_{1/2})$

Spaces of sequences l^p, c, c_0

D1. For $1 \le p < \infty$ we define l^p as the space of sequences summable with p-th power and equipped with the norm

$$\left\|(x_k)_{k=1}^\infty\right\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$$

For $p = \infty$, we define l^∞ as complex-valued bounded sequences with the norm

$$\left\|(x_k)_{k=1}^\infty\right\|_p = \sup_{k\in\mathbb{N}} |x_k|.$$

Justify briefly that l^p is a Banach space.

D2. (**Schauder basis for l^p**) Consider l^p where $1 \leq p < \infty$ and unit vectors $e_i = (0, 0, ..., 0, 1, 0, ...)$ where 1 is on the i-th coordinate. Prove that for any $x = (x_1, x_2, ...) \in l_p$, $\sum_{i=1}^n x_i e_i \to x$ converges in l_p, i.e. that

$$\left\| \sum_{i=1}^n x_i e_i - x \right\|_p \to 0.$$

We say that system $\{e_i\}_{i\in\mathbb{N}}$ is a Schauder basis of l_p. How the situation changes for $p = \infty$?

D3. Consider space of real-valued sequences $(x_0, x_1, x_2, ...)$ such that $\lim_{k\to\infty} x_k$ exists and equip it with a supremum norm, i.e. $\|(x_k)_{k=1}^\infty\|_\infty = \sup_k |x_k|$. Prove that this is a Banach space (it is usually denoted with c).

D4. Similarly, consider subspace of c of sequences $(x_0, x_1, x_2, ...)$ converging to 0 equipped with supremum norm (it is usually denoted with c_0). Prove that it is a Banach space.

D5. (**Schauder basis for c_0**) Consider problem D2. with space c. More precisely, given unit vectors $e_i = (0, 0, ..., 0, 1, 0, ...)$ where 1 is on the i-th coordinate, prove that for any $x = (x_1, x_2, ...) \in c_0$, $\sum_{i=1}^n x_i e_i \to x$ converges in c_0, i.e. that

$$\left\| \sum_{i=1}^n x_i e_i - x \right\|_\infty \to 0.$$

Note once again, that according to Exercise D2., this is not the case for l^∞ but c_0 is a closed subset of l^∞.

D6. Consider subset of c of sequences converging to 1. Can this subset be a normed space (no matter how the norm is defined)?

D7. Consider set l^1 with l^∞ norm. Is it a normed space? Is it a Banach space?

Additional problems

E1. (**Minkowski functional**) Let $(E, \|\cdot\|_E)$ be a normed space and $C \subset E$ be an open and convex subset with $0 \in C$. For every $x \in E$ we define:

$$\rho(x) = \inf \left\{ \alpha > 0 : \frac{x}{\alpha} \in C \right\}.$$

Prove the following properties of ρ which is called Minkowski functional of C:

(a) $\rho(\lambda x) = \lambda \rho(x)$ for any $\lambda > 0$ and $x \in E$,

(b) $\rho(x + y) \leq \rho(x) + \rho(y)$ for any $x, y \in E$,

(c) there is a constant M so that $0 \leq \rho(x) \leq M\|x\|_E$ for all $x \in E$,

(d) $C = \{x \in E : \rho(x) < 1\}$.

E2. (**Minkowski functional as a norm**) Under assumptions of Problem E1., suppose additionally that C is symmetric i.e. $C = -C$ and bounded. Prove that ρ defines a norm on E which is equivalent to the norm $\|\cdot\|_E$.

E3. (**generalized Hölder inequality**) Let $f_i \in L^{p_i}$ for $i = 1, ..., n$ where $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}$. Prove that $f_1 f_2 ... f_n \in L^p$. More precisely, prove the bound

$$\|f_1 f_2 ... f_n\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} ... \|f_n\|_{p_n}.$$

Hint: one can simplify to the case $p = 1$, then proceed by induction.

E4. (**generalized Minkowski inequality**) Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be two measure spaces. Let $F : X \times Y \to \mathbb{R}$ be a measurable and nonnegative map. Prove that

$$\left| \int_Y \left| \int_X F(x,y) d\mu(x) \right|^p d\nu(y) \right|^{\frac{1}{p}} \leq \int_X \left| \int_Y |F(x,y)|^p d\nu(y) \right|^{\frac{1}{p}} d\mu(x).$$

Deduce from this standard Minkowski inequality.