Functional Analysis (WS 20/21), Problem Set 11

(compact operators, spectral theory)

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Let E, F be Banach spaces. Let $T: E \to F$ be a linear operator. We say that T is compact if $\overline{T(B_1(0))}$ is compact in F. Equivalently, if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, there is a convergent subsequence in $\{Tx_n\}_{n\in\mathbb{N}}$.

Compact operators

A1. Prove that if $T: E \to F$ is compact then T is bounded.

A2. Prove that the following are equivalent

- (A) $\overline{T(B(0,1))}$ is compact,
- (B) if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, there is a convergent subsequence in $\{Tx_n\}_{n\in\mathbb{N}}$.

A3. Prove that if $T: E \to F$ and $S: E \to F$ are compact then $T + S$ is compact.

A4. Let $g \in C[0,1]$ and $T: C[0,1] \to C[0,1]$ be defined with the formula $Tf(x) = \int_0^x f(t)g(t)dt$. Prove that T is a compact operator.

A5. Let E be infinite dimensional Banach space. Prove that identity operator on E is not compact.

A6. Let E be infinite dimensional Banach space. Prove that if $T: E \to E$ is compact then $I - T$ is not compact.

A7. Prove that the identity operator $I: C^\alpha[0,1] \to C[0,1]$ is compact.

A8. Let $p \geq q \geq 1$. Prove that the identity operator $I: L^p(0,1) \to L^q(0,1)$ is not compact. *Hint*: Consider oscillating sequence $f_n(x) = \sin(2\pi n x)$.

A9. Let $K \in L^2(\Omega \times \Omega)$ be a measurable kernel on $\Omega \times \Omega$. We define Hilbert-Schmidt operator $T: L^2(\Omega) \to L^2(\Omega)$ with

$$Tf(x) = \int_\Omega K(x,y)f(y)\,dy.$$

Apply Banach-Alaoglu-Bourbaki Theorem in the separable Hilbert space $L^2(\Omega)$ to deduce that if $K \in L^2(\Omega \times \Omega)$ then T is a compact operator.[1]

Spectral theorem for compact operators on Hilbert spaces

Theorem (Fredhold-Riesz): Let H be an infinite dimensional Hilbert space and $T: H \to H$ be a compact operator. Then, $0 \in \sigma(T)$, all non-zero $\lambda \in \sigma(T)$ are eigenvalues of T and 0 can be the only limit point of $\sigma(T)$.

B1. Prove that if $T: H \to H$ is compact and H is infinite dimensional then $0 \in \sigma(T)$.

B2. Find $\sigma(T)$ where $T: L^2(0,1) \to L^2(0,1)$ is given with the formula $Tf(x) = \int_0^x f(y)dy$.

[1] It was proved in the lecture that Hilbert-Schmidt operator is compact for continuous kernel K.

B3. Find all bounded sequences $(y_i)_{i\in\mathbb{N}}$ such that $T : l^2 \to l^2$ defined with $Tx = (x_i\, y_i)_{i\in\mathbb{N}}$ is compact. *Hint:* Recall what is $\sigma(T)$.

B4. Find all bounded and continuous functions $g : \mathbb{R} \to \mathbb{R}$ such that the multiplication operator $G : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined with $Gf(x) = g(x)\, f(x)$ is compact. *Hint:* Recall what is $\sigma(G)$.

Spectral theorem for self-adjoint compact operators

Theorem (Hilbert-Schmidt): Let H be a separable Hilbert space and $A : H \to H$ be a compact self-adjoint operator. Then, there is a countable orthonormal basis of H consisting of eigenvectors of A. Moreover, the corresponding eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ are real. If $\dim H = \infty$, $\lambda_k \to 0$ as $k \to \infty$.

C1. **(roots of operators)** Let $A : H \to H$ be self-adjoint and compact linear operator on a separable Hilbert space H. Let $n \in \mathbb{N}$. Prove that there exists a a bounded linear operator $B : H \to H$ such that $B^n = A$.

C2. **(approximate inverse)** Let $A : H \to H$ be a self-adjoint and compact linear operator on a separable Hilbert space H. Suppose that $\ker A = \{0\}$. Prove that there exists a sequence of operators $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n A x \to x$ as $n \to \infty$.