Functional Analysis (WS 20/21), Problem Set 12

(convolutions, density of smooth functions and Schwartz spaces)

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For $f, g \in L^1(\mathbb{R}^d)$ we define convolution $f * g$:

$$f * g(x) = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy.$$

Clearly $f*g = g*f$. Convolutions are also studied for functions defined on subsets $\Omega \subset \mathbb{R}^d$ assuming one takes care about appropriate domain of definition for f and g.

Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a standard mollifier (approximate identity) i.e. for a smooth radial nonnegative function $\eta \in C^\infty$ supported on the ball $B_1(0)$ such that $\int_{\mathbb{R}^d} \eta(x)\, dx = 1$ we let $\eta_\varepsilon(x) = \varepsilon^{-d}\eta(x/\varepsilon)$. Then, for all $f \in L^p(\mathbb{R}^d)$ we have $f * \eta_\varepsilon \to f$ in $L^p(\mathbb{R}^d)$ and a.e. when $\varepsilon \to 0$. As $f * \eta_\varepsilon$ is smooth, standard truncation argument shows that $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$ (lecture). Similar results hold for $L^p(\Omega)$ as any $f \in L^p(\Omega)$ can be extended to $f \in L^p(\mathbb{R}^n)$.

Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of infinitely differentiable functions such that the family of seminorms

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty$$

where $\alpha, \beta \in \mathbb{N}^d$. We usually say that Schwartz space consists of functions vanishing faster than any polynomial. We say that $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ if $p_{\alpha,\beta}(f_n - f) \to 0$ as $n \to \infty$ for all $\alpha, \beta \in \mathbb{N}^d$.

Convolutions

A1. Let $g \in C_c^k(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$. Prove that $f * g$ is $C^k(\mathbb{R}^n)$. Find the formulas for the derivatives of $f * g$.

A2. (**Young's inequality**) Prove Young's convolutional inequality: if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ then $f * g \in L^r(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, 1 \leq p, q, r \leq \infty$. Moreover,

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

A3. Let $f \in L^1(\mathbb{R}^n)$ and g be Lipschitz bounded function. Prove that $f * g$ is again a bounded and Lipschitz function.

A4. Let g be a symmetric bounded function i.e. $g(x) = g(-x)$. Prove that for $f, h \in L^1(\mathbb{R}^d)$ we have:

$$\int_{\mathbb{R}^d} f(x)\, g * h(x) = \int_{\mathbb{R}^d} f * g(x)\, h(x).$$

Density of smooth functions

B1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Prove that $f * \eta_\varepsilon \to f$ uniformly on compact subsets of $\mathbb{R}^n$.

B2. Let $f \in L^p(\mathbb{R})$. Obtain explicit (in terms of ε) bounds for $\partial^k(f * \eta_\varepsilon)$ in $L^p(\mathbb{R})$ and $L^\infty(\mathbb{R})$.

B3. Let $f \in L^1(\Omega)$. Prove that if $\int_\Omega f\, \varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$ then $f = 0$ a.e. in Ω.

B4. Let $f \in L^1(\Omega)$. Prove that if $\int_\Omega f\,\varphi \geq 0$ for all $\varphi \in C_c^\infty(\Omega), \varphi \geq 0$ then $f \geq 0$ a.e. in Ω.

B5. Let $f \in L^1(0,1)$. Prove that if $\int_0^1 f\,\varphi' = 0$ for all $\varphi \in C_c^\infty(0,1)$ then f is constant a.e. in $(0,1)$.

B6. (**cutoff**) Let Q be a d-dimensional cube with side length 1. Prove that there is a smooth function f such that $f = 1$ on Q, $f = 0$ on $\mathbb{R}^n \setminus 2Q$ and $f \in [0,1]$.

B7. (**smooth partition of unity**) Write $\mathbb{R}^d = \cup_{i=1}^\infty Q_i$ where Q_i are d-dimensional cubes with side length 1. Prove that there exists f_i such that f_i is supported on $2\,Q_i$, $f_i \in [0,1]$ and $\sum_{i=1}^\infty f_i = 1$.

B8. Let $1 < p < \infty$. Prove that $f_n \rightharpoonup f$ in $L^p(\Omega)$ if and only if

$$\int_\Omega f_n\,\varphi \to \int_\Omega f\,\varphi \text{ for all } \varphi \in C_c^\infty(\Omega) \text{ and } \{f_n\}_{n\in\mathbb{N}} \text{ is bounded in } L^p(\Omega).$$

B9. Are smooth functions dense in $L^\infty(0,1)$?

Schwartz space

S1. $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

S2. For $a > 0$, $e^{-a\|x\|^2} \in \mathcal{S}(\mathbb{R}^d)$.

S3. If $f \in \mathcal{S}(\mathbb{R}^d)$ then $f \in L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.