Functional Analysis (WS 20/21), Problem Set 2

(operators and their norms)

Compiled on 01/11/2020 at 2:07pm

For the linear map $T : X \to Y$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces, we define norm of the operator T as
$$\|T\| = \sup_{x : \|x\|_X \leq 1} \|Tx\|_Y.$$
If $\|T\| < \infty$ we say T is a bounded linear operator. We write $\mathcal{L}(X, Y)$ for the space of all bounded linear operators between X and Y. We write $X^* = \mathcal{L}(X, \mathbb{R})$ for the dual space of X i.e. space of bounded and linear operators $X : E \to \mathbb{R}$ (or $X : E \to \mathbb{C}$ in the case of complex linear spaces) called bounded functionals.

In case $(Y, \|\cdot\|_Y)$ is additionally assumed to be a Banach space, $\mathcal{L}(X, Y)$ equipped with the operator norm turns out to be a Banach space too.

Basic facts on operator norm and $\mathcal{L}(X, Y)$

A1. Check that
$$\sup_{x : \|x\|_X \leq 1} \|Tx\|_Y = \sup_{x : \|x\|_X = 1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$
so that there are three equivalent definition of the operator norm.

A2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Let $T : X \to Y$ be a linear map. Check that the following conditions are equivalent:

(a) T is a bounded linear operator,

(b) T is continuous at 0,

(c) T is continuous,

(d) T is Lipschitz continuous (i.e. $\|Tx_1 - Tx_2\|_Y \leq C \|x_1 - x_2\|_X$ for some constant C).

A3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Justify briefly that $\mathcal{L}(X, Y)$ equipped with operator norm is a normed space.

A4. **(bound on composition)** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Let $T : X \to Y$ and $S : Y \to Z$ be bounded linear operators. Check that $\|S \circ T\| \leq \|S\| \, \|T\|$.

Computation of the operator/functional norms

B1. Let $E = \{u \in C([0, 1]) : u(0) = 0\}$. Justify briefly that E with a supremum norm is a Banach space. Consider a linear functional $\varphi : E \to \mathbb{R}$ defined with $\varphi(u) = \int_0^1 u(t)\, dt$. Prove that $\varphi \in E^*$ and compute norm $\|\varphi\|$ of this functional. Is there $u \in E$ such that $\|u\|_\infty \leq 1$ and $\varphi(u) = \|\varphi\|$?

B2. Consider a linear functional $\varphi : C[0, 1] \to \mathbb{R}$ defined with $\varphi(u) = \int_0^{\frac{1}{2}} u(t)\, dt - \int_{\frac{1}{2}}^1 u(t)\, dt$. Prove that $\varphi \in (C[0, 1])^*$ and compute norm $\|\varphi\|$ of this functional. Is there $u \in C[0, 1]$ such that $\|u\|_\infty \leq 1$ and $\varphi(u) = \|\varphi\|$?

B3. Consider functional $\varphi : l_1 \to \mathbb{R}$ defined with $\varphi(u) = \sum_{i=1}^{\infty} \frac{1}{2^n} u_n$ where $u = (u_1, u_2, u_3, ...)$. Prove that $\varphi \in (l_1)^*$ and compute its norm $\|\varphi\|$.

B4. More generally, fix $v \in l^{\infty}$ where $v = (v_1, v_2, v_3, ...)$. Let $\varphi_v(u) = \sum_{i=1}^{\infty} v_n\, u_n$ where $u = (u_1, u_2, u_3, ...)$. Prove that $\varphi_v \in (l_1)^*$ and compute its norm $\|\varphi_v\|$ in terms of v.

B5. Consider functional $\varphi : c_0 \to \mathbb{R}$ defined with $\varphi(u) = \sum_{i=1}^{\infty} \frac{1}{2^n} u_n$ where $u = (u_1, u_2, u_3, ...)$. Compute norm $\|\varphi\|$ of this functional (i.e. $\sup_{u : \|u\|_{\infty} \leq 1} |\varphi(u)|$) and prove that $\varphi \in (c_0)^*$. Is there $u \in c_0$ such that $\|u\|_{\infty} \leq 1$ and $\varphi(u) = \|\varphi\|$?

B6. For $u \in C[0,1]$ we set $\varphi(u) = f\left(\frac{1}{2}\right)$. Prove that $\varphi \in (C[0,1])^*$ and compute its norm.

B7. Let μ be a Borel measure on $[0,1]$ and $\varphi_{\mu}(f) = \int_0^1 f(x)\, \mathrm{d}\mu(x)$. Prove that $\varphi \in (C[0,1])^*$ and compute its norm in terms of μ. Compare with Problem B6.

B8. Let $\varphi : C[0,1] \to \mathbb{R}$ be a linear functional such that $\varphi(f) \geq 0$ whenever $f \geq 0$. Prove that $\varphi \in (C[0,1])^*$ and compute its norm. Compare with Problems B6. and B7.

B9. (**averaging operators**) Let $1 \leq p \leq \infty$ and $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ be defined with

$$T(f)\,(y) = \frac{1}{\lambda_n(B(y,1))} \int_{B(y,1)} f(x)\, \mathrm{d}x$$

where $B(y,1)$ denotes a unit ball with center $y \in \mathbb{R}^d$. Check that T is well - defined (i.e. it is L^p-valued), prove that it is a bounded linear operator and compute its norm.

B10. (**discrete derivative**) Let $1 \leq p \leq \infty$ and $T : l^p \to l^p$ be defined with

$$T\left((a_n)_{n \geq 1}\right) = (a_{n+1} - a_n)_{n \geq 1}.$$

Check that T is well - defined (i.e. it is l^p-valued), prove that it is a bounded linear operator and compute its norm.

B11. Let $1 \leq p \leq \infty$ and consider operator $T : l^p \to l^p$ defined with $T\left((a_n)_{n \geq 1}\right) = \left(\frac{a_n}{n+1}\right)_{n \geq 1}$. Decide whether T is a well-defined bounded operator and if yes, compute its norm.

B12. Let $1 \leq p \leq \infty$ and consider operator $T : L^p(0,1) \to L^p(0,1)$ defined with $(Tf)(x) = f(x^2)$. Prove that $T \in \mathcal{L}(L^p(0,1), L^p(0,1))$ and compute its norm.

B13. (**multiplication operator**) Let $1 \leq p \leq \infty$, $g \in L^{\infty}(X, \mathcal{F}, \mu)$ and consider operator $T : L^p(X, \mathcal{F}, \mu) \to L^p(X, \mathcal{F}, \mu)$ defined with $(Tf)(x) = f(x)\, g(x)$. Prove that $T \in \mathcal{L}(L^p, L^p)$ and compute its norm.

B14. Let $T : (C^1[0,1], \|\cdot\|_{\infty}) \to (C[0,1], \|\cdot\|_{\infty})$ be defined with $Tf = f'$. Is T a bounded operator from $(C^1[0,1], \|\cdot\|_{\infty})$ to $(C[0,1], \|\cdot\|_{\infty})$?

<u>More on linear functionals</u>

C1. Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Prove that any linear functional on X is bounded.

C2. Let $(X, \|\cdot\|)$ be an infinite dimensional normed space. Prove that there is a linear functional on X that is not bounded.

C3. In infinite dimensional spaces even "projections" can be discontinuous. To see this consider space of polynomials $\mathcal{P}[0,1]$ equipped with L^1 norm on $[0,1]$.

a. Let φ_0 be a functional defined on $\mathcal{P}[0,1]$ with

$$\varphi_0(a_0 + a_1\,x + a_2\,x^2 + ... + a_n\,x^n) = a_0.$$

Is $\varphi_0 \in (\mathcal{P}[0,1], \|\cdot\|_1)^*$? *Hint:* consider $\varphi_0((x-1)^n)$.

b. More generally, let φ_k be a functional defined on $\mathcal{P}[0,1]$ with

$$\varphi_k(a_0 + a_1\,x + a_2\,x^2 + ... + a_k\,x^k + ... + a_n\,x^n) = a_k.$$

Is $\varphi_k \in (\mathcal{P}[0,1], \|\cdot\|_1)^*$?

Remark: This pathology will not happen in Hilbert spaces.

C4. Let $(E, \|\cdot\|_E)$ be a normed space and $\varphi : E \to \mathbb{R}$ be a linear functional on E.

(a) Prove that if $\varphi \neq 0$, then there is a one dimensional subspace $F \subset E$ such that

$$E = \ker\varphi \oplus F$$

i.e. for all $x \in E$, there are uniquely determined $y \in \ker\varphi$ and $z \in F$ such that $x = y + z$.

(b) Prove that $\varphi \in E^*$ (i.e. it is bounded) if and only if its kernel is closed in E.

Completeness of $\mathcal{L}(X, Y)$

D1. (**dual spaces are always Banach**) Let $(X, \|\cdot\|_X)$ be a normed space. Justify briefly that the dual space X^* of bounded linear functionals on X is a Banach space.

D2. **(extension from dense subset)** Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Suppose that D is a dense linear subspace of X and $T : (D, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a bounded linear operator. Prove that T has a unique bounded extension

$$\widetilde{T} : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

such that

$$Tx = \widetilde{T}x \text{ for } x \in D \qquad \text{and} \qquad \|T\|_{\mathcal{L}(D,Y)} = \|\widetilde{T}\|_{\mathcal{L}(X,Y)}.$$

Hint: If $x \in X \setminus D$, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$.

D3. (**exponential of the operator**) Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Let $T : X \to Y$ be a bounded operator. Check that the series $\sum_{k=0}^{\infty} \frac{T^k}{k!}$ converges in $\mathcal{L}(X, Y)$. This is usually definition of e^T. Note that this generalizes e^A for $A \in \mathbb{R}^{n\times n}$.

D4. (**inverse operator**) Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Let $T : X \to Y$ be a bounded operator such that $\|T\| < 1$. Check that the series $\sum_{k=0}^{\infty} T^k$ converges in $\mathcal{L}(X, Y)$.