Functional Analysis (WS 20/21), Problem Set 4

(Closed Graph Theorem and Inverse Mapping Theorem)

Compiled on 23/11/2020 at 11:36am

Closed Graph Theorem Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ a linear operator. Then, T is bounded if and only if its graph

$$G(T) = \{(x, Tx) : x \in X\}$$

is closed in the product space $X \times Y$.

Inverse Mapping Theorem Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator that is bijective. Then, T^{-1} is also bounded.

Open Mapping Theorem(*) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator that is surjective. Then, T is open i.e. there is a constant $c > 0$ such that

$$B_Y(0, c) \subset T(B_X(0, 1)).$$

Closed Graph Theorem

A1. (**Inverse Mapping Theorem**) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces. Suppose that $T \in \mathcal{L}(X, Y)$ such that

$$\text{for all } y \in Y \text{ there is a unique } x \in X \text{ such that } Tx = y$$

i.e. T is bijective. Prove that $T^{-1} \in \mathcal{L}(Y, X)$.

A2. Let $(X, \|\cdot\|_X)$ be a Banach space. Consider a linear operator $T : X \to X^*$ such that for all $x, y \in X$:

$$(Tx)(y) = (Ty)(x)$$

Prove that T is a bounded operator. *Hint:* Let $x_n \to x$ and consider $(Tx_n)(z)$.

A3. Let $(X, \|\cdot\|_X)$ be a Banach space. Consider a linear operator $T : X \to X^*$ such that for all $x \in X$:

$$(Tx)(x) \geq 0.$$

Prove that T is a bounded operator.

A4. Show that, up to an equivalence of norms, the supremum norm is the only norm on $C[0, 1]$ which makes $C[0, 1]$ complete and which also implies the pointwise convergence.

A5. Show that, up to an equivalence of norms, the $\|\cdot\|_p$ norm is the only norm on $L^p(0, 1)$ which makes $L^p(0, 1)$ complete and which also implies pointwise converges a.e. of some subsequence.

A6. Let $(X, \|\cdot\|_X)$ be a Banach space and $P : X \to X$ be a linear operator such that

- the kernel of P is closed in X,
- the image of P is closed in X,
- $P(P(x)) = P(x)$ for all $x \in X$.

Prove that $P \in \mathcal{L}(X, X)$.

A7. Let $X = C[0,1]$, $Y = C^{2020}[0,1]$. We define operator $T : X \to Y$ as follows. If $f \in X$, $Tf = x_f$ is the solution of the following ODE

$$x_f^{(2020)}(t) + t\, x_f^{(2019)}(t) + t^2\, x_f^{(2018)}(t) + \ldots t^2 019\, x'_f(t) + t^{2020}\, x_f(t) = f(t)$$
$$x_f^{(i)}(0) = 0 \text{ for } i = 0, \ldots, 2019.$$

- Are X, Y Banach spaces with their natural norms?
- Is T well-defined and linear?
- Is $T \in \mathcal{L}(X, Y)$?

Inverse Mapping Theorem

B1. Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X and suppose that X is a Banach space with respect to both of them. Finally, suppose that $\|x\|_1 \leq C\|x\|_2$ for some constant C. Then, there is a constant c such that

$$\|x\|_2 \leq c\|x\|_1$$

and hence, both norms are equivalent on X.

B2. Prove that $C[0,1]$ equipped with $L^p(0,1)$ norm is not a Banach space for $1 \leq p < \infty$.

B3. Prove that the space of summable sequences l^1 equipped with l^∞ norm is not a Banach space.

B4. Let $1 \leq p < \infty$. Prove that $L^p(0,1)$ equipped with $L^q(0,1)$ norm for $1 \leq q < p$ is not a Banach space.

B5. Let $(E, \|\cdot\|_E)$ be a Banach space and $A : E \to E$ a bounded linear operator. Suppose that there is a natural number $n \in \mathbb{N}$ and real numbers $c_1, \ldots, c_n$ such that

$$I + c_1 A + \ldots + c_n A^n = 0,$$

where I is the identity operator. Prove that A^{-1} exists and it is a bounded linear operator.

Open Mapping Theorem

C1. Let $X = (l_1, \|\cdot\|_1)$ and $Y = (l_1, \|\cdot\|_\infty)$. Prove that identity operator $T : X \to Y$ defined with $Tx = x$ is a bounded linear map that is not open. Why OMT does not hold in this case?