**Functional Analysis (WS 20/21), Problem Set 5**

**(Introduction to Hilbert Spaces)**

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## Basic properties of Hilbert spaces

A1. Define a scalar product on:

- $L^2(0,1)$ over $\mathbb{R}$ and over $\mathbb{C}$.
- $l^2$ over $\mathbb{R}$ and over $\mathbb{C}$.

A2. Verify that $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$ defines an inner product on $C[0,1]$ over $\mathbb{R}$. Is $(C[0,1], \langle\cdot,\cdot\rangle)$ a Hilbert space?

A3. (**Pythagorean Theorem**) Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $x, y \in H$ such that $x$ is perpendicular to $y$. Prove that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

A4. Is the space of measurable functions $f : (0,1) \to \mathbb{R}$ such that $\int_0^1 |f(t)|^2\, e^t\, dt$ a Hilbert space? What is the inner product?

A5. We say that a Banach space $(X, \|\cdot\|)$ is uniformly convex if for any $\epsilon > 0$, there is $\delta > 0$ such that for any $x, y \in E$:

$$\text{if } \|x\| = \|y\| = 1 \text{ and } \|x-y\| \geq \epsilon \text{ then } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

Prove that any Hilbert space is uniformly convex.[1]

A6. (**Bessel's inequality**) Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal sequence in Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Prove that for any $x \in H$

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

A7. For $1 \leq p \leq \infty$ consider space $L^p(0,1)$. Prove that norm $\|\cdot\|_p$ satisfies parallelogram identity:

$$2\,\|x\|_p^2 + 2\,\|y\|_p^2 = \|x+y\|_p^2 + \|x-y\|_p^2$$

if and only if $p = 2$. *Hint:* Consider functions with disjoint supports.

[1]A deep result due to Milman and Pettis asserts than any uniformly convex Banach space $E$ is reflexive, i.e. $E^{**} = E$ up to an isometric isomorphism. This somehow connects *geometric* and *analytical* properties of Banach spaces. Note that this is still weaker than Riesz Representation Theorem asserting that for any Hilber space $H$ we have $H = H^*$.

## Orthogonal complements and projections

1. Let $K \subset H$ be non-empty, convex and closed subset of Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. We know that for every $f \in H$, there exists a unique $P_K(f) \in K$ called **projection onto $K$** such that

$$\inf_{g \in K} \|f - g\| = \|f - P_K(f)\|.$$

The map $P_K : H \to H$ is a bounded linear operator with $\|P_K\| = 1$.

2. For an arbitrary subset $K \subset H$ we define its **orthogonal complement**

$$K^\perp = \{x \in H : \langle x, v \rangle = 0 \text{ for all } v \in K\}.$$

3. Let $M \subset H$ a closed subspace. There is a decomposition

$$H = M \oplus M^\perp$$

i.e. for any $f \in H$ we can write $f = f_M + f_{M^\perp}$ where $f_M \in M$ and $f_{M^\perp} \in M^\perp$ are uniquely determined[a]. More precisely,

$$f_M = P_M f, \qquad f_{M^\perp} = f - P_M f = P_{M^\perp} f.$$

[a] This is no longer true in Banach spaces. See example with $c_0 \subset l^\infty$ where $c_0$ (which is closed linear subspace of $l^\infty$) has no complement in $l^\infty$: https://math.stackexchange.com/questions/132520/complement-of-c-0-in-ell-infty

All Hilbert spaces below are assumed (for simplicity) to be real.

B1. Let $K \subset H$. Prove that $K^\perp$ is a closed linear subspace of $H$.

B2. Consider
$$X = \{f \in L^2(0,1) : f(x) = 0 \text{ for all } x \in [0, 1/2]\}$$
as a subspace of $L^2(0,1)$ (over $\mathbb{R}$).

(a) Is $X$ closed in $L^2(0,1)$?

(b) Find $X^\perp$.

(c) Find decomposition $f = P_X f + P_{X^\perp} f$.

B3. Consider
$$X = \{f \in L^2(-1,1) : f(x) = f(-x)\}$$
as a subspace of $L^2(-1,1)$ (over $\mathbb{R}$).

(a) Is $X$ closed in $L^2(-1,1)$?

(b) Find $X^\perp$.

(c) Find decomposition $f = P_X f + P_{X^\perp} f$.

B4. Find a real polynomial $w(t)$ of degree at most 2 such that $\int_0^1 |w(t) - t^4|^2 dt$ is the smallest.

B5. Find a real polynomial $w(t)$ of degree at most 1 such that $\int_0^1 |w(t) - \sqrt{t}|^2 dt$ is the smallest.

B6. Let

$$E = \left\{ f \in L^2(0,1) : \int_0^1 f(t)\,\mathrm{d}t = \int_0^1 f(t)\,t\,\mathrm{d}t = 0 \right\}$$

Compute

$$\inf_{f \in E} \int_0^1 |t^3 - f(t)|^2.$$

B7. Let

$$E = \left\{ f \in L^2(0,1) : \int_{-1}^1 f(t)\,\mathrm{d}t = \int_{-1}^1 f(t)\,t\,\mathrm{d}t = 0 \right\}$$

Let $g(x) = \frac{1}{1+x^2}$. Find $P_E\, g$, $P_{E^\perp}\, g$ and distance of $g$ from $E$.

## Riesz Representation Theorem

C1. Prove that there exists a function $f \in L^2(0,1)$ such that $\int_0^1 t^2 f(t)dt = \int_0^1 e^t f(t)dt$. Is this function uniquely determined (say, up to some scalings)?

C2. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space. Suppose that $a : H \times H \to \mathbb{R}$ is a symmetric bilinear continuous form that is coercive, i.e. there is a constant $\beta$ such that $a(u,u) \geq \beta \|u\|^2$. Prove that $(H, a(\cdot, \cdot))$ is a Hilbert space with the same topology as $(H, \langle \cdot, \cdot \rangle_H)$ (i.e. norms are equivalent).

C3. **(Lax-Milgram Lemma, simplified version)** Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space. Suppose that $a : H \times H \to \mathbb{R}$ is a symmetric bilinear continuous form that is coercive, i.e. there is a constant $\beta$ such that $a(u,u) \geq \beta \|u\|^2$. Let $l \in H^*$. Prove that there is a unique $u \in H$ such that

$$a(u,v) = l(v) \text{ for all } v \in H.$$

C4. Let $H$ be a Hilbert space, $\varphi \in H^*$ and $T \in \mathcal{L}(H,H)$. Prove or disprove: there exists uniquely determined $y \in H$ such that for all $x \in H$ we have $\varphi(Tx) = \langle x, y \rangle$.