Functional Analysis (WS 20/21), Problem Set 7

(introduction to weak convergence)

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Let $(E, \|\cdot\|)$ be a Banach space. We say that sequence $(x_n)_{n\geq 1} \subset E$ converges weakly to $x \in E$ if for every $\varphi \in E^*$ we have $\varphi(x_n) \to \varphi(x)$. We write $x_n \rightharpoonup x$.

General properties

A1. Write explicitly, using representation theorems, what does it mean to converge weakly in L^p (for $1 \leq p < \infty$) and H where H is a Hilbert space.

A2. Prove that weak limits are unique: if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ then $x = y$.

A3. Prove that sequences converging weakly are bounded, i.e. if $x_n \rightharpoonup x$ then there is a constant C such that $\|x_n\| \leq C$ where C does not depend on $n \in \mathbb{N}$. Moreover, prove the bound

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

A4. Prove that if $x_n \to x$ then $x_n \rightharpoonup x$.

A5. Prove that if $x_n \rightharpoonup x$ and $f_n \to f$ in E^* then $f_n(x_n) \to f(x)$ as $n \to \infty$.

A6. **(Mazur Lemma)** Let $C \subset E$ be a convex set. Prove that C is closed for convergence in norm if and only if C is closed for weak convergence. *Hint*: One implication is trivial, for another use Hahn-Banach.

C is closed for convergence in norm if for any $\{x_n\}_{n\geq 1}$ such that $x_n \to x$ it follows that $x \in C$. This is exactly the same as statement that C is closed in E.
C is closed for weak convergence if for any $\{x_n\}_{n\geq 1}$ such that $x_n \rightharpoonup x$ it follows that $x \in C$.

A7. Let $(H, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Prove that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ and $x_n \rightharpoonup 0$.

A8. (**Banach-Alaoglu-Bourbaki Theorem, special case**) Let $(H, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H. Prove that $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging weakly to some $x \in H$. [1]

Particular examples

B1. In l^2 consider sequence of unit vectors $\{e_n\}$. Decide if this sequence converges strongly or weakly in l^2. If yes, what is the limit?

B2. Prove that $\sin(nx) \rightharpoonup 0$ but $\sin^2(nx) \rightharpoonup \frac{1}{2}$ in $L^p(0, 2\pi)$ for $1 < p < \infty$. Hence, nonlinearities do not preserve weak limits. *Remark:* Unfortunately, one can show much more: if F is a function such that $F(x_n) \rightharpoonup F(x)$ for all $x_n \rightharpoonup x$, then F is an affine function.

B3. Let $f_n(x) = \sin(nx)$. Decide if $f_n(x)$ converges strongly in $L^2(0, 2\pi)$, weakly in $L^2(0, 2\pi)$ or a.e. on $(0, 2\pi)$.

[1] This result is probably the most important one in Functional Analysis and holds for more general spaces.

B4. Let H be a Hilbert space. Suppose that $x_n \rightharpoonup x$ in H and $\limsup_{n\to\infty} \|x_n\|^2 \leq \|x\|^2$. Prove that $x_n \to x$ in H.

B5. (A) Let $f : (0,1) \to \mathbb{R}$ be a bounded function and extend it periodically to the function $f : \mathbb{R} \to \mathbb{R}$. Prove that $f_n(x) = f(n\,x)$ converges weakly in $L^2(0,1)$ to $\overline{f} = \int_0^1 f(y)\,\mathrm{d}y$ (i.e. constant function).

(B) Find weak limits in $L^2(0,1)$ of sequences $\sin(2\pi n x)$ and $\sin^2(2\pi n x)$.