Functional Analysis (WS 20/21), Problem Set 8

(orthonormal systems and basis in Hilbert spaces)

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We say that the set $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ is an **orthonormal system** if $\|e_\alpha\| = 1$ and $\langle e_{\alpha_1}, e_{\alpha_2}\rangle = 0$. Any orthonormal system satisfies **Bessel's inequality**: for each $x \in H$ we have

$$\sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle^2 \leq \|x\|^2.$$

The sum above contains at most countably many nonzero terms.

We say that the set $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ is an **orthonormal basis** of H if there is no bigger orthonormal set $\{e_\beta\}_{\beta\in\mathcal{B}}$ such that $\{e_\alpha\}_{\alpha\in\mathcal{A}} \subset \{e_\beta\}_{\beta\in\mathcal{B}}$, i.e. $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ is a maximal orthonormal set in H. Any orthonormal basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ satisfies:

(A) **Parseval's identity** (i.e. equality in Bessel's inequality) for all $x \in H$ we have

$$\sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle^2 = \|x\|^2.$$

(B) for each $x \in H$ we can wrie $x = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle\, e_\alpha$ and this series is convergent in H.

Below, we obtain easier characterizations of orthonormal basis (see Problems 1 and 2).

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1. (**criterion 1**) Prove that the orthonormal set $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ is actually the basis if and only if
$$\langle x, e_\alpha\rangle = 0 \text{ for all } \alpha \in \mathcal{A} \implies x = 0.$$
2. (**criterion 2**) Prove that the orthonormal set $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ is actually the basis if and only if
$$H = \overline{\mathrm{span}(e_\alpha : \alpha \in \mathcal{A})},$$
where span is the space of all linear combinations.
3. Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal Schauder basis of $L^2(0,1)$. For given $t \in [0,1]$ compute:
$$\sum_{n=1}^{\infty} \left| \int_0^t x^3 f_n(x)\, dx \right|^2.$$
4. Let $a_n = \int_0^{2\pi} t^2\, e^{int}\, \mathrm{d}t$. Compute $\sum_{n\in\mathbb{Z}} |a_n|^2$ and $\sum_{n=1}^{\infty} |a_n|^2$.
5. Prove that if H is separable than it has a countable orthonormal basis.
6. Prove that separable infinite dimensional Hilbert space H is isometrically isomorphic to l^2.
7. Consider Radamacher system: $r_0 = 1$ and $r_n = \mathrm{sgn}(\sin(2^n \pi t))$. Prove that this is an orthogonal system but it is not an orthogonal basis of $L^2(0,1)$.
8. If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonomal set in H, for all $x \in H$ we have $\langle x, e_n\rangle \to 0$ when $n \to \infty$.

9. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal set in Hilbert space H. Consider operator $T : H \to c_0$ defined with
$$Tx = \left(\frac{n}{n+1}\langle x, e_n\rangle\right)_{n\in\mathbb{N}}.$$
Prove that T is well-defined. Is T a bounded linear operator? If yes, compute its norm.

10. If H is infinite dimensional, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ and $x_n \rightharpoonup 0$.[1]

11. Let $y \in l^\infty$, $\{e_n\}_{n\in\mathbb{N}}$ is an orthonomal set in H and $u_n = \frac{1}{n}\sum_{i=1}^n e_i\, y_i$. Prove that $u_n \to 0$.

12. Let $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ be an orthonormal basis of H. Prove that
$$x_n \rightharpoonup x \text{ in } H \iff \langle x_n - x, e_\alpha\rangle \to 0 \text{ for all } \alpha \in \mathcal{A}.$$

13. (**Banach-Alaoglu-Bourbaki Theorem, special case**) Let H be a separable Hilbert space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H. Prove that $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging weakly to some $x \in H$. [2] [3]

14. Let μ be a gaussian measure on $\mathbb{R}$, i.e. a measure with density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Let $H = L^2(\mathbb{R}, \mu)$.
    (A) Let $X = \operatorname{span}(1, x, x^2)$. Prove that X is a linear subspace of H.
    (B) Recall Gram-Schmidt algorithm. Use it to find an orthonormal basis of X.
    (C) Compute the distance of $f(x) = |x|$ from X.

    When evaluating integrals you may use certain moments of normally distributed random variables without a proof.

15. Consider two σ-finite measure spaces (X, μ) and (Y, ν). Prove or disprove:
    (A) If $\{f_j\}_{j\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ are orthonormal sets in $L^2(X, \mu)$ and $L^2(Y, \mu)$ respectively, then the set
    $$f_j \otimes g_k : X \times Y \to \mathbb{R}, \qquad f_j \otimes g_k(x, y) := f_j(x)\, g_k(y), \qquad k, j \in \mathbb{N}$$
    is an orthonormal set of $L^2(X \times Y, \mu \otimes \nu)$. Here, $\mu \otimes \nu$ denotes product measure on $X \times Y$.
    (B) If $\{f_j\}_{j\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ form orthonormal basis of $L^2(X, \mu)$ and $L^2(Y, \mu)$ respectively, then the set $f_j \otimes g_k$ is an orthonormal basis of $L^2(X \times Y, \mu \otimes \nu)$.

16. There are Hilbert spaces that are not separable. Consider space of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ only for countably many $x \in \mathbb{R}$ equipped with the scalar product
$$\langle f, g\rangle = \sum_{x\in\mathbb{R}} f(x)\, g(x).$$
Prove that it is a Hilbert space but it is not separable (find uncountable set of elements x_i such that $\|x_i - x_j\| \geq 1$).

[1] This is copied from Problem Set 7.
[2] This result is probably the most important one in Functional Analysis and holds for more general spaces.
[3] This is copied from Problem Set 7.