Hyperbolic Conservation Laws (WS 20/21)

Homeworks

Compiled on 06/01/2021 at 5:01pm

Problems have to be solved and the solutions have to be e-mailed to

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before the class begins (16:00). As the subject of your e-mail use

CL-**n**-homework

where **n** is the number of the submitted homework (**n** is 1 for the first one).

Contents

Homework 1 (for 28/10/2020)

Let $G \in L^1(\mathbb{R}^n)$, $\mathcal{B}$ be a bounded subset of $L^p(\mathbb{R}^n)$ and $1 \le p < \infty$. Let $\mathcal{F}$ be a family of functions defined as

$$\mathcal{F} = \{G * B : B \in \mathcal{B}\}.$$

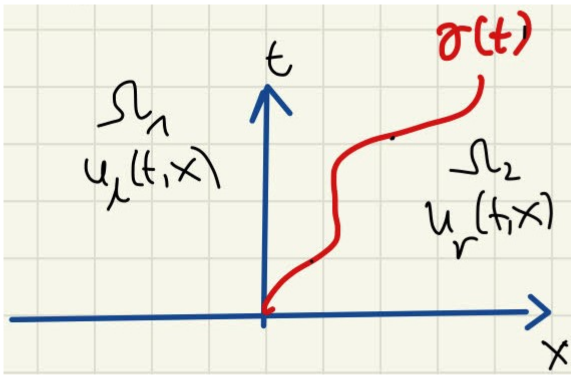
Prove that $\mathcal{F}|_\Omega$ has a compact closure in $L^p(\Omega)$ for any $\Omega \subset \mathbb{R}^n$ with finite measure.

Homework 2 (for 4/11/2020)

Let $\gamma(t) : \mathbb{R}^+ \to \mathbb{R}$ be a curve of class C^1 and

$$u(t,x) = \begin{cases} u_l(t,x), & x < \gamma(t), \\ u_r(t,x), & x > \gamma(t), \end{cases}$$

as in the figure below.



We assume that u_l, u_r are pointwise solutions to the conservation law

$$u_t + \mathrm{div} F(u) = 0 \tag{1}$$

in Ω_1, Ω_2 respectively. Prove that u is an entropy solution to (1) if and only if

$$\dot{\gamma}(t)\ [\eta(u_r(t,\gamma(t))) - \eta(u_l(t,\gamma(t)))] \geq [Q(u_r(t,\gamma(t))) - Q(u_l(t,\gamma(t)))]$$

for all entropy/entropy-flux pairs (η, Q) for all convex η. This can be seen as a Rankine-Hugoniot version of entropy inequality.

Homework 3 (for 12/11/2020)

Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ and $u_0 \in L^\infty(\mathbb{R}^d)$. Suppose that u is an entropy solution to the scalar conservation law

$$u_t + \mathrm{div} F(u) = 0$$

with initial condition u_0, i.e. for all entropy/entropy-flux pairs (η, Q) with η convex and all test functions $\varphi \in C^1_c([0,\infty) \times \mathbb{R}^d)$ such that $\varphi \geq 0$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(t,x)\, \eta(u(t,x))\, \mathrm{d}x\, \mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot Q(F(u(t,x)))\, \mathrm{d}x\, \mathrm{d}t + \\ + \int_{\mathbb{R}^d} \varphi(0,x)\, \eta(u_0(x))\, \mathrm{d}x \geq 0. \tag{2}$$

Prove that u is a distributional solution, i.e. for all test functions $\varphi \in C^1_c([0,\infty) \times \mathbb{R}^d)$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(t,x)\, u(t,x)\, \mathrm{d}x\, \mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot F(u(t,x))\, \mathrm{d}x\, \mathrm{d}t + \\ + \int_{\mathbb{R}^d} \varphi(0,x)\, u_0(x)\, \mathrm{d}x = 0. \tag{3}$$

This shows that entropy solutions can be defined directly by the entropy inequality (2) without assuming distributional identity (3).

Homework 4 (for 26/11/2020)

Let $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be a bounded (i.e. $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$) distributional solution to the scalar conservation law

$$u_t + \operatorname{div}(F(u)) = 0$$

with bounded initial condition u_0 and locally Lipschitz continuous F. Prove that $t \mapsto u(t, x)$ has weakly* continuous for all $t \geq 0$.

Next, consider equation studied in the class

$$u_t + \operatorname{div}(F(u)) = \mu$$

where μ is a locally bounded measure and specify reasonable assumptions on μ so that $t \mapsto u(t, x)$ has weakly* continuous for all $t \geq 0$.

Hint: Follow the reasoning from the class and observe that if a measure term is zero, you can use Arzela-Ascoli Theorem.

Homework 5 (for 9/12/2020)

Here are two examples of Young measures that one can compute directly.

(A) Let $u : (0,1) \to \mathbb{R}$ be given with $u(x) = \mathbb{1}_{(0,1/2)}(x) - \mathbb{1}_{(1/2,1)}(x)$. Extend u periodically to the whole of $\mathbb{R}$ and define $u_j : (0,1) \to \mathbb{R}$ with $u_j(x) = u(jx)$. Prove that the Young measure $\{\nu_x\}_{x\in(0,1)}$ of the sequence $\{u_j\}_{j\in\mathbb{N}}$ is given by

$$\nu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1.$$

As this Young measure does not depend on x, we say that the Young measure $\{\nu_x\}_{x\in(0,1)}$ is homogeneous.

(B) Consider functions $u_j : (0,1) \to \mathbb{R}$ defined with $u_j(x) = \sin(2\pi jx)$. Prove that the Young measure $\{\nu_x\}_{x\in(0,1)}$ of the sequence $\{u_j\}_{j\in\mathbb{N}}$ is absolutely continuous with respect to the Lebesgue measure on $(-1,1)$ and its density equals

$$\frac{1}{\pi\sqrt{1-y^2}}.$$

(C) Use result in (B) to compute weak limits of $\sin(2\pi jx)$, $\sin^2(2\pi jx)$ and $\sin^3(2\pi jx)$.

Hint: It is sufficient to study $\int_0^1 \varphi(x)\, h(u_j(x))\, \mathrm{d}x$ for sufficiently many test functions φ and h.

Homework 6 (for 16/12/2020)

Let $\Omega = (0,1)^d$ be a unit cube and consider homogenization problem

$$-\mathrm{div}(A(x/\varepsilon)\,\nabla u^\varepsilon) = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

where $f \in L^2(\Omega)$, $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ and A is extended periodically to the whole of $\mathbb{R}^n$. We assume that A is strongly elliptic in the sense that for all $\xi \in \mathbb{R}^n$ we have

$$\lambda^{-1}\,|\xi|^2 \le \xi^T A(y)\,\xi \le \lambda\,|\xi|^2.$$

(A) Prove that for $\varepsilon \in (0,1)$ there is the unique solution $u^\varepsilon \in H^1_0(\Omega)$.

(B) Prove that there is some $\xi_0 \in L^2(\Omega)$ such that up to a subsequence,

$$\nabla u^\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega)$$
$$u^\varepsilon \to u \text{ strongly in } L^2(\Omega)$$
$$A(x/\varepsilon)\,\nabla u^\varepsilon \rightharpoonup \xi_0 \text{ weakly in } L^2(\Omega)$$
$$A(x/\varepsilon) \rightharpoonup \overline{A} = \int_\Omega A(y)\,\mathrm{d}y \text{ weakly in } L^2(\Omega)$$

(C) Prove that $\nabla u^\varepsilon A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup \nabla u\,\xi_0$ in $L^1(\Omega)$.

Remark: In fact, one can use more complicated tricks to see that $\xi_0 = \overline{A}\,\nabla u$.

Homework 7 (for 8/01/2021)

We know from tutorials that when $\Omega \subset \mathbb{R}^n$ is a bounded subset and $\{\mu_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is compact in $W^{-1,q}(\Omega)$ for $1 \leq q < \frac{n}{n-1}$.

Argue in a similar manner to prove that if a sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, it is compact in $W^{-1,q}(\Omega)$ for $1 \leq q < p^*$ where p^* is a usual Sobolev exponent.

Homework 8 (for 8/01/2021)

Here is another result of compensated compactness type proved similarly to the div-curl lemma. Let $u_n, v_n : (0,T) \times \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is a bounded subset. Assume that

$$\begin{aligned} \{v_n\}_{n\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;H_0^1(\Omega)), \\ \{u_n\}_{n\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)), \\ \{\partial_t u_n\}_{n\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega)), \end{aligned}$$

where time derivatives above are understood in the sense of distributions. Suppose that

$$u_n \rightharpoonup u,\ v_n \rightharpoonup v \text{ in } L^2((0,T)\times\Omega).$$

Prove that $u_n\, v_n \to u\, v$ in the sense of distributions (in duality with smooth compactly supported functions).

Hint: As always, start by writing $u_n = -\Delta z_n$ for some z_n.

Homework 9 (for 20/01/2021)

(A) (revision) Let $h(x) = \gamma\,\mathbb{1}_{[a,b]}$. Find explicitly the derivative h' i.e. a linear functional on the Schwartz space $\mathcal{S}(\mathbb{R})$ such that

$$h'(\varphi) = -\int_{\mathbb{R}} h(x)\,\varphi'(x)\,\mathrm{d}x.$$

Show that $h' \in \mathcal{M}(\mathbb{R})$.

(B) Let $u^\varepsilon : \Omega \to \mathbb{R}$ be a sequence of bounded functions such that $u^\varepsilon \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$. Suppose that the Young measure generated by $\{u^\varepsilon\}_{\varepsilon>0}$ is given by

$$\mu_x = \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_2 + \frac{1}{4}\delta_5$$

for a.e. $x \in \Omega$. Draw the corresponding kinetic function (i.e. weak* limit of $\chi(\xi, u^\varepsilon(x))$ in $L^\infty(\Omega \times \mathbb{R})$).

(C) Let $v^\varepsilon : \Omega \to \mathbb{R}$ be a sequence of bounded functions such that $v^\varepsilon \overset{*}{\rightharpoonup} v$ in $L^\infty(\Omega)$. Suppose that its kinetic function is $f(x,\xi) = \mathbb{1}_{[0,1]}(\xi)$. Find Young measure generated by sequence $\{v^\varepsilon\}_{\varepsilon>0}$.