Hyperbolic Conservation Laws Tutorial

Topic 1: Introduction. Examples of CL. Essentials from mathematical analysis.

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1. Examples of HCL, scalar and vector case.

In this class, we want to study eqns of the form

$$\partial_t u + \operatorname{div} F(u) = 0.$$

Here, $u : \underset{t}{\mathbb{R}^+} \times \underset{x}{\mathbb{R}^n} \longrightarrow \mathbb{R}^m$ so $u = (u_1, \dots, u_m)$ where

$u_i : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

Moreover, $F : \mathbb{R}^m \to M^{m \times n}$ where $M^{m\times n}$ are matrices with m rows and n columns.

$$F(u) = \begin{pmatrix} F_{11}(u) & \dots & F_{1m}(u) \\ \vdots & & \\ F_{m1}(u) & \dots & F_{mn}(u) \end{pmatrix}$$

Divergence is computed wrt each row separately.

$$\operatorname{div} F(u) = \begin{pmatrix} \partial_{x_1} F_{11}(u) & \dots & \partial_{x_n} F_{1n}(u) \\ \vdots & & \vdots \\ \partial_{x_1} F_{m1}(u) & \dots & \partial_{x_n} F_{mn}(u) \end{pmatrix}.$$

Example 1: Scalar equation ($x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $u \in \mathbb{R}$).

We will develop a whole theory for such equations.

Example 2: Systems of equations. ($u \in \mathbb{R}^m$). This remains unsolved even in the case $x \in \mathbb{R}$, $t \in \mathbb{R}^+$.

Example 3: Euler's equation in one space dimension :

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + (\rho v^2 + p)_x = 0 \\ (\rho E)_t + (\rho E v + p v)_x = 0 \end{cases}$$

Where ρ is density, v is velocity, E is energy density (per unit mass).

2A. Mollification, convergence results.

(Ref: Brezis, 4.4, Mollification)

Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a standard mollifier, i.e.

- $\eta \geq 0$,
- $\int \eta = 1$,
- η is compactly supported in $B(0,1)$,
- η is smooth.

For $\varepsilon > 0$, we define $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right)$ so that $\int \eta_\varepsilon = 1$, η_ε is supported in the ball $B(0, \varepsilon)$.

Three properties that are crucial in PDEs:

1. If $f \in L^1_{loc}(\mathbb{R}^n)$, $f * \eta_\varepsilon$ is smooth and
$$D^k(f * \eta_\varepsilon) = f * D^k \eta_\varepsilon.$$

2. If $f \in C(\mathbb{R}^n)$, $f * \eta_\varepsilon \to f$ uniformly on compact subsets of $\mathbb{R}^n$.

3. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ then $f * \eta_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$.

<u>SLN to 1:</u> Note that $(f * \eta_\varepsilon)(x) = \int f(y) \eta_\varepsilon(x-y)\, dy$ so differentiation does not see f. $\square$

SLN to 2:

$$|f * \eta_\varepsilon(x) - f(x)| = \left| \int f(x-y)\eta_\varepsilon(y)\,dy - \int f(x)\eta_\varepsilon(y)\,dy \right|$$

$$\leq \int \eta_\varepsilon(y)\,|f(x-y) - f(x)| = \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,|f(x-y) - f(x)|\,dy$$

Let $x \in K$ (compact set) and let K_ε be closure of convex hull of K. As f is continuous,

$$\forall_{\gamma > 0}\ \exists_{\varepsilon_\gamma}\ \forall_{x,y \in K_\varepsilon}\ |x-y| \leq \varepsilon_\gamma \Rightarrow |f(x) - f(y)| \leq \gamma.$$

Hence, when $\varepsilon \leq \varepsilon_\gamma$, we have

$$|f * \eta_\varepsilon(x) - f(x)| \leq \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy \cdot \gamma = \gamma.$$

As γ is arbitrary and independent of $x \in K$, this concludes the proof.

SLN to 3: Let $\gamma > 0$. There is $f_0 \in C_c(\mathbb{R}^n)$ such that

$\|f_0 - f\|_{L^p(\mathbb{R}^n)} \leq \gamma$. We write

$$\|f - f * \eta_\varepsilon\|_{L^p} \leq \|f - f_0\|_p + \|f_0 - f_0 * \eta_\varepsilon\|_p + \|(f_0 - f) * \eta_\varepsilon\|_p$$

By Young's ineq: $\|(f_0 - f) * \eta_\varepsilon\|_p \leq \|f_0 - f\|_p$.

By 2, $\limsup_{\varepsilon \to 0} \|f - f * \eta_\varepsilon\|_p \leq 2\|f_0 - f\|_p \leq 2\gamma$

As γ is arbitrary, the proof is concluded. $\square$.

2B. Strong compactness in L^p $(1 \le p < \infty)$: Kolmogorov – Riesz – Frechet theorem.

(Ref: Brezis, Chapter 4.5)

In infinite-dimensional spaces, bounded sets do not have to be compact as the following example shows.

1. Let H be inf. dim. Hilbert space. Let $\{e_i\}_{i\in\mathbb{N}}$ be its orthonormal basis. Prove that $\{e_i\}_{i\in\mathbb{N}}$ is bounded but not compact.

SLN: Clearly $\|e_i\| = 1$ so the set is bounded. Moreover,

$$\|e_i - e_j\|^2 = \langle e_i - e_j, e_i - e_j\rangle = \|e_i\|^2 + \|e_j\|^2 - 2\langle e_i, e_j\rangle = 2$$

so $\{e_i\}$ cannot have a convergent subsequence. $\square$

There are two ways to overcome this difficulty in PDEs:

- additional conditions on bounded sets
- working in weak topologies.

In this class, we gonna work on PDEs of the form

$\partial_t u + \operatorname{div} F(u) = 0$ so we need strong compactness.

nonlinearity

First, we recall classical A-A theorem.

THEOREM (Arzela, Ascoli). Let (K,d) be a compact metric case and let $\mathcal{H}$ be a bounded set of $C(K)$ (wrt $\|\cdot\|_\infty$ norm). Assume that $\mathcal{H}$ is uniformly continuous

$$\forall_{\varepsilon>0} \ \exists_{\delta_\varepsilon} \ d(x_1,x_2) \le \delta_\varepsilon \Rightarrow |f(x_1) - f(x_2)| \le \varepsilon \quad \forall_{f\in\mathcal{F}}$$

Then, the closure of $\mathcal{H}$ in $C(K)$ is compact.

Now, to formulate L^p analogue we need some notation.
For $f: \mathbb{R}^n \to \mathbb{R}$ we define $T_h f(x) = f(x+h)$ $(h \in \mathbb{R}^n)$.
If $\mathcal{F}$ is a family of functions on $\mathbb{R}^n$, we write $\mathcal{F}|_\Omega$ for the family of their restrictions to $\Omega \subset \mathbb{R}^n$.

THEOREM (Kolmogorov, Riesz, Frechet)

Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^n)$ where $1 \leq p < \infty$. Assume that

$$\lim_{|h| \to 0} \| T_h f - f \|_p = 0 \quad \text{uniformly in } f \in \mathcal{F}$$

(i.e. $\forall_{\varepsilon > 0} \exists_{\delta_\varepsilon > 0} \quad |h| \leq \delta_\varepsilon \Rightarrow \| T_h f - f \|_p \leq \varepsilon \quad \forall_{f \in \mathcal{F}}$)

Then, $\mathcal{F}|_\Omega$ has compact closure in $L^p(\Omega)$ for each $\Omega \subset \mathbb{R}^n$ of finite measure.

(2) How to see AA in Kolmogorov theorem?

$\| T_h f - f \|_p^p = \int | f(x+h) - f(x) |^p \, dx$ where $|(x+h) - x| \leq h$.

This is "equicontinuity in L^p."

(3) The theorem is formulated for $\mathcal{F} \subset L^p(\mathbb{R}^n)$. How to get compactness of $\mathcal{F} \subset L^p(\Omega)$?

We can extend functions to be defined on $\mathbb{R}^n$. The result is formulated in this way so that the shift operator is well-defined.

(4) The result gives compactness of $\mathcal{F}|_{\Omega}$ whenever Ω has a finite measure.

Example: fix $\varphi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \varphi \subset [0,1]$. Let $f_n(x) = \varphi(x+n)$. Prove that:

(A) $\{f_n\}$ is bounded in $L^p(\mathbb{R})$.

(B) $\{f_n\}$ satisfies L^p-equicontinuity condition.

(C) $\{f_n\}$ is **NOT** compact in $L^p(\mathbb{R})$.

Ad. (A): $\int_{\mathbb{R}} |f_n|^p = \int_{\mathbb{R}} |\varphi(x+n)|^p = \int_{\mathbb{R}} |\varphi(x)|^p$ — ind. of n.

Ad. (B): $\|T_h f_n - f_n\|_p^p = \int_{\mathbb{R}} |f_n(x+h) - f_n(x)|^p dx =$

$$= \int_{\mathbb{R}} |\underbrace{\varphi(x+h+n)}_{\text{supported in } [-n-h,\,-n-h+1]} - \underbrace{\varphi(x+n)}_{\text{supported in } [-n,\,-n+1]}|^p dx = (*)$$

So the length of support is $|(-n+1) - (-n-h)| = |1+h|$.

$(*) \leq |1+h| \, \|\varphi'\|_\infty^p \cdot h^p \longrightarrow 0$ as $|h| \to 0$ independ. of n.

Ad. (C): Suppose it is. In particular, there is a subseq. converging a.e. As φ is supported on $[0,1]$, the subseq.

converges to 0. It follows that the subsequence $\{f_{n_k}\}$ should converge to 0 in L^p. But $\int |f_{n_k}|^p = \int |e|^p \not\to 0$.

$\square$.

We will see later that there is an additional condition that guarantee compactness in $L^p(\mathbb{R}^n)$.

We move to the proof of Kolmogorov Theorem:

5 Follow steps to prove Kolmogorov Theorem:

STEP 1: let $\{\rho_{1/n}\}$ be a standard mollifier. Fix $\varepsilon > 0$ and let δ_ε be as above. We claim

$$\| \rho_{1/n} * f - f \|_p \leq \varepsilon \qquad \forall_{f \in \mathcal{F}} \ \forall_{n > \frac{1}{\delta_\varepsilon}}$$

PROOF: $\int | \rho_{1/n} * f(x) - f(x) |^p dx =$

$$= \int \left| \int \rho_{1/n}(y) f(x-y)\, dy - \int \rho_{1/n}(y) f(x)\, dy \right|^p dx$$

$$= \int \left| \int \rho_{1/n}(y) [f(x-y) - f(x)]\, dy \right|^p dx \leq$$

Hölder with measure $\sim \rho_{1/n}$

$$\leq \int \int \rho_{1/n}(y) |f(x-y) - f(x)|^p\, dy\, dx =$$

$$= \int \rho_{1/n}(y) \underbrace{\int |f(x-y) - f(x)|^p dx}_{\leq \varepsilon^p \text{ as } |y| \leq \frac{1}{n} \leq \delta_\varepsilon} dy \leq \varepsilon^p. \checkmark$$

$|y| \leq \frac{1}{n}$

STEP 2: We have inequalities:

- $\|\varrho_{1/n} * f\|_\infty \leq C_n \|f\|_p$ (C_n – depends on n and p but p is fixed)
- $|\varrho_{1/n} * f(x_1) - \varrho_{1/n} * f(x_2)| \leq C_n \|f\|_p |x_1 - x_2|$.

PROOF: By Young's convolutional inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

we have $\|\varrho_{1/n} * f\|_\infty \leq \|f\|_p \underbrace{\|\varrho_{1/n}\|_{p'}}_{C_n.}$

To see the second inequality, we need to estimate

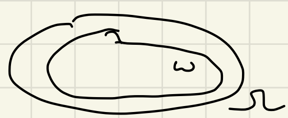
$\|\nabla(\varrho_{1/n} * f)\|_\infty = \|(\nabla \varrho_{1/n}) * f\|_\infty$ by properties of conv.

Again, by Young's ineq.,

$$\|(\nabla \varrho_{1/n}) * f\|_\infty \leq \|f\|_p \underbrace{\|\nabla \varrho_{1/n}\|_{p'}}_{C_n.} \qquad \checkmark$$

STEP 3: Let $\Omega \subset \mathbb{R}^n$ be a set of finite measure. Let $\varepsilon > 0$. Then, there is $\omega \subset \Omega$ such that

$$\|f\|_{L^p(\Omega \setminus \omega)} \leq \varepsilon \qquad \forall_{f \in \mathcal{F}}$$



("lack of concentration").

PROOF: By triangle inequality,

$$\|f\|_{L^p(\Omega\setminus\omega)} \leq \|f - f * \rho_{1/n}\|_{L^p(\Omega\setminus\omega)} + \|f * \rho_{1/n}\|_{L^p(\Omega\setminus\omega)}$$

$$\leq \|f - f * \rho_{1/n}\|_{L^p(\mathbb{R}^n)} + \|f * \rho_{1/n}\|_{L^\infty} |\Omega\setminus\omega|.$$

$$\leq \varepsilon + C_n \cdot \|f\|_p \, |\Omega\setminus\omega| \quad \text{by Step 1 and Step 2.}$$

where n is chosen so that $n > \frac{1}{\delta_\varepsilon}$.

Choosing ω sufficiently large, we conclude the proof (we use here that $\mathcal{F}$ is bounded in L^p).

STEP 4: $\mathcal{F}|_\Omega$ is totally bounded (i.e. $\forall_{\varepsilon>0}$ there are $N(\varepsilon)$ balls of radius ε covering $\mathcal{F}|_\Omega$).

[in complete metric spaces: total boundedness is equivalent to compactness of the closure].

PROOF: Fix $\varepsilon > 0$. Choose ω from Step 3. By Step 2, the set $\{\rho_{1/n} * f\}_{f\in\mathcal{F}} =: \mathcal{H}_n$ ($n \geq \frac{1}{\delta_{\varepsilon/3}}$ fixed) has compact closure in $C(\bar{\omega})$. By dominated convergence, $\mathcal{H}_n$ has also compact closure in $L^p(\bar{\omega})$, i.e. there are $g_1, \dots, g_{N_\varepsilon}$ such that

$$\forall_{f\in\mathcal{F}} \; \exists_i \; \|\rho_{1/n} * f - g_i\|_{L^p(\bar{\omega})} \leq \frac{\varepsilon}{3}$$

We need to cover $\mathcal{F}|_\Omega$. Note that we already have covering of $\mathcal{H}_n$ in $L^p(\bar{\omega})$ norm.

Let $\tilde{g}_i = \begin{cases} g_i & \text{on } \overline{\omega} \\ 0 & \text{on } \Omega\setminus\overline{\omega}. \end{cases}$ and we claim that

$B(\tilde{g}_i, \varepsilon)$ cover $\mathcal{F}|_\Omega$ in $L^p(\Omega)$ norm. Indeed, let $f \in \mathcal{F}|_\Omega$, choose g_i such that

$$\| \rho_{1/n} * f - g_i \|_{L^p(\overline{\omega})} \leq \varepsilon/3.$$

As g_i is 0 on $\Omega\setminus\overline{\omega}$, we have

$$\| f - g_i \|_{L^p(\Omega)} \leq \| f \|_{L^p(\Omega\setminus\overline{\omega})} + \| f - g_i \|_{L^p(\overline{\omega})} \leq$$

$$\leq \| f \|_{L^p(\Omega\setminus\overline{\omega})} + \| f - f * \rho_{1/n} \|_{L^p(\overline{\omega})} + \| f * \rho_{1/n} - g_i \|_{L^p(\overline{\omega})}$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\square$

6 Under additional equitightness condition,

$$\forall_{\varepsilon>0} \; \exists_{\Omega \text{ bdd}} \quad \| f \|_{L^p(\mathbb{R}^n\setminus\Omega)} \leq \varepsilon \quad \forall f \in \mathcal{F}$$

prove compactness of $\mathcal{F}$ in $L^p(\mathbb{R}^n)$ (without restriction).

SLN: We modify Step 3 above. This time we take

$$\tilde{g}_i = \begin{cases} g_i & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases}$$

$\square$

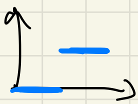
⑦ [Connection with Reillich-Kondrachov thm]

It is standard result in PDEs that $W^{1,p}(U)$ is compactly embedded in $L^p(U)$ $1 \leq p \leq \infty$ ($p=\infty$ is Arzela-Ascoli).

Recall that if $D_i^h u(x) = \dfrac{u(x+he_i) - u(x)}{h}$ then

- $\|D_i^h u\|_p \leq C \|u\|_{W^{1,p}}$ $(1 \leq p < \infty)$
- $\|D_i^h u\|_p \leq C$ (ind of h) $\Rightarrow u \in W^{1,p}$ $(1 < p < \infty)$.

(Ref: Evans, 8.5.2)

(There is easy example for $p=1$: ).

So, if $u \in W^{1,p}(U) \Rightarrow \|D_i^h u\|_{L^p} \leq C \|u\|_{W^{1,p}} \Rightarrow$
$\|T_h u - u\|_{L^p} \leq Ch \|u\|_{W^{1,p}} \longrightarrow 0$ uniformly if we are in bounded set in $W^{1,p}$.

Kolmogorov theorem is stronger because it implies compactness from weaker estimates like

$$\|T_h u - u\|_{L^p} \leq C h^{1/2} \|u\|_{W^{1,p}}$$

or

$$\|T_h u - u\|_{L^p} \leq C f(h)$$

where f is a modulus of continuity.

In particular, in the proof of existence of solutions to conservation laws, we gonna deal with the 2nd case.

8 [HOMEWORK]

Let $G \in L^1(\mathbb{R}^n)$, $\mathcal{B}$ be a bounded subset of $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. Let $\mathcal{F} := \{ G * B : B \in \mathcal{B} \}$. Prove that $\mathcal{F}|_{\Omega}$ has compact closure in $L^p(\Omega)$ for any Ω with finite measure.

2C. Weak compactness in L^p $(1 \leq p < \infty)$: Banach–Alaoglu theorem.

We say that $\{f_n\} \subset L^p(\Omega)$, $1 \leq p < \infty$ converges WEAKLY to $f \in L^p(\Omega)$ if $\int f_n \phi \to \int f \phi \quad \forall \phi \in L^{p'}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$.

We say that $\{f_n\} \subset L^\infty(\Omega)$ converges WEAKLY* to $f \in L^\infty(\Omega)$ if $\int f_n \phi \to \int f \phi \quad \forall \phi \in L^1(\Omega)$.

<u>THEOREM.</u> (Banach, Aloglu, Bourbaki)

Let $1 < p < \infty$ and let $\{f_n\} \subset L^p(\Omega)$ be a bounded sequence in $L^p(\Omega)$. Then, there is a subsequence converging weakly in $L^p(\Omega)$.

Similarly, if $\{f_n\} \subset L^\infty(\Omega)$ is a bounded seq. in $L^\infty(\Omega)$, there is a subsequence converging weakly-* in $L^\infty(\Omega)$.