Hyperbolic Conservation Laws Tutorial

Topic 2: Some properties of entropy solutions.

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CONTENTS:

1. Characterization theorem for Riemann problem.

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Two most common notions of solutions to

$$u_t + \operatorname{div}(F(u)) = 0 \qquad \left(u : \mathbb{R}^n \to \mathbb{R},\ F(u) : \mathbb{R} \to \mathbb{R}^n \right)$$

It is typical to assume that F is uniformly convex (i.e. $F'' \geq C > 0$).

1) Integral (distributional) $\to$ roughly speaking equation has to be satisfied a.e. and R-H conditions have to be satisfied.

2) Entropy solution: integral and

$$(*) \quad \eta(u)_t + \operatorname{div}(Q(u)) \leq 0$$

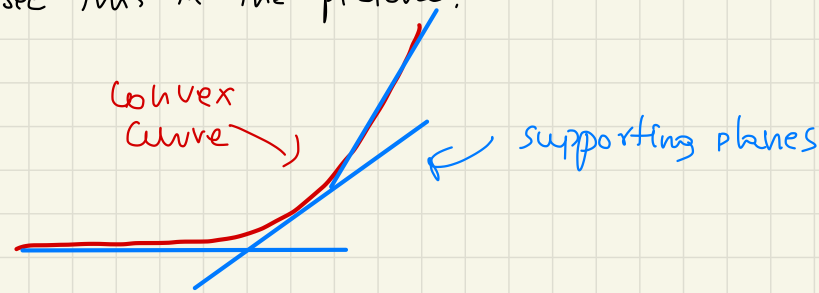
for all pairs (η, Q) s.t. $\eta : \mathbb{R} \to \mathbb{R}$, $Q : \mathbb{R} \to \mathbb{R}^n$ $DQ = D\eta \cdot DF = \eta' \cdot DF$ and η convex.

Some remarks:

1) Since we work with scalar equation, we have a lot of entropies η. For systems condition $DQ = D\eta \cdot DF$ is usually overdetermined but some systems have entropy!

2) Usually Kruzkhov entropies $\eta_k(u) = |u-k|^+$ with flux $Q_k(u) = \operatorname{sgn}(u-k)\,(F(u) - F(k))$ are sufficient in the sense that if $(*)$ is satisfied for all (η_k, Q_k), $k \in \mathbb{R}$ then $(*)$ is satisfied for all (η, Q) with η convex. (at least for locally bounded solutions)

It's hard to write this argument explicitly so maybe its easier to see this in the picture.



A little bit more rigorously, for η convex we write

$$\eta(u) \approx c + \sum_{k=1}^{n} \alpha_i |u - c_i|^+$$

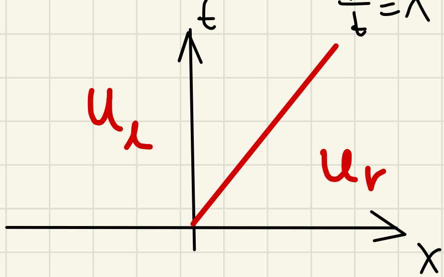
And very rigorously: fix $\varepsilon > 0$ and divide $[0, U]$ for subintervals of length $\leq \varepsilon$. On each of them find supporting plane. In each point take max over all planes.

3) For Riemann problem i.e. with initial data

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

R-H condition reads $\lambda = \dfrac{F(u_l) - F(u_r)}{u_l - u_r}$

$$\frac{x}{t} = \lambda$$

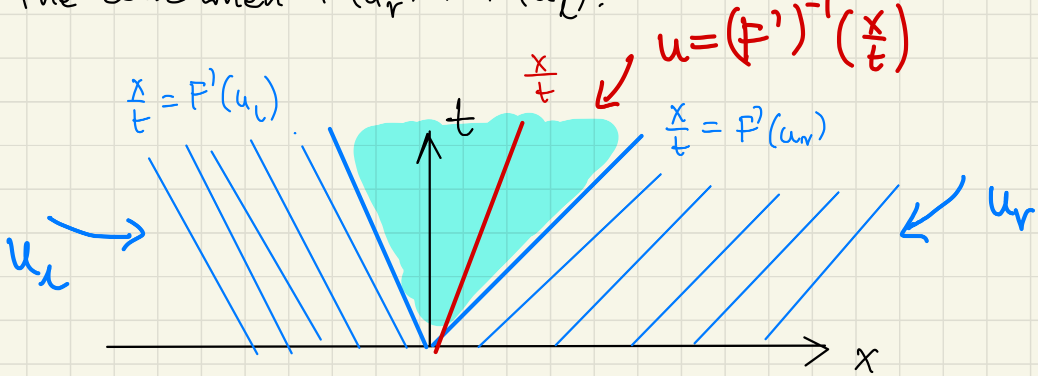


and we can construct solution

$$u(t,x) = \begin{cases} u_l & x/t \geq \lambda \\ u_r & x/t < \lambda . \end{cases}$$

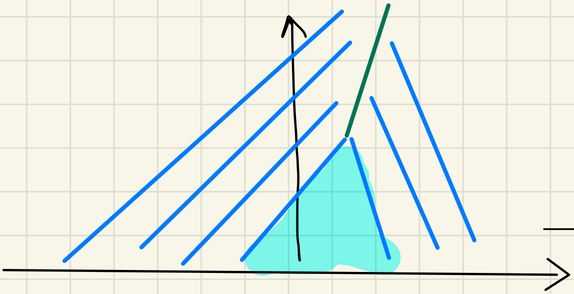
This is called **SHOCK**.

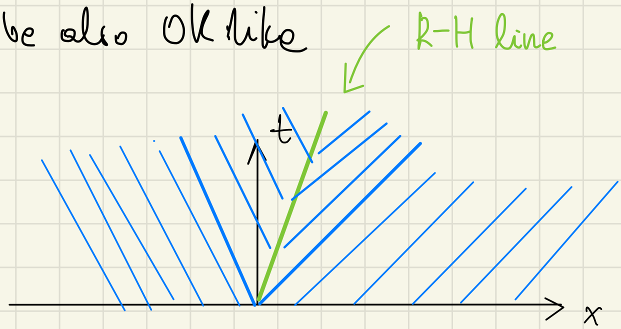
4) We may also have **RAREFACTION WAVES**. This is usually the case when $F'(u_r) > F'(u_l)$.



This works because characteristic equations for CL has speed $F'(u_0(x))$ (think about $u_t + \operatorname{div} F(u)$ as about transport equation). It was checked in the lecture that this is a solution.

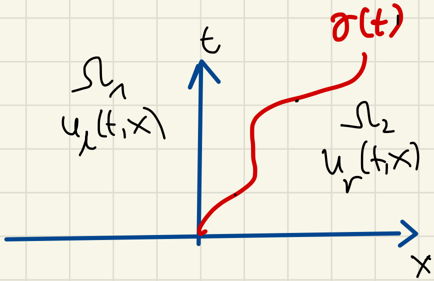
5) Other possibilites may be also OK like





Next week, we will prove that entropy solutions are unique.

Now, our target is to characterize entropy solutions in the scalar case.



<u>LEMMA</u> (R-H for entropy condition)

Suppose that $[0,\infty) \times \mathbb{R} = \Omega_1 \cup \Omega_2$.

Let u be as in the picture $u = \begin{cases} u_1 & \Omega_1 \\ u_2 & \Omega_2 \end{cases}$

and u_1, u_2 solve $u_t + F(u)_x = 0$ <u>pointwisely</u> in Ω_1, Ω_2 respectively. Then, u is an entropy solution iff

$$\dot{\gamma}(t) \left[\eta\left(u_r(t,\gamma(t))\right) - \eta\left(u_l(t,\gamma(t))\right)\right] \geq$$

$$\geq Q\left(u_r(t,\gamma(t))\right) - Q\left(u_l(t,\gamma(t))\right).$$

(Remark: this can be generalized for more curves $\gamma_1, \gamma_2, \dots$).

<u>PROOF</u>: Homework for next week.

<u>THEOREM 1</u> Let $u_r < u_l$. Then, $u(t,x) = \begin{cases} u_l & x < \lambda t \\ u_r & x > \lambda t \end{cases}$

is an entropy solution where $\lambda = \dfrac{F(u_l) - F(u_r)}{u_l - u_r}$.

<u>PROOF</u>: u given by the formula is an entropy solution iff

$$\lambda \left[\eta(u_r) - \eta(u_l)\right] \geq Q(u_r) - Q(u_l)$$

due to lemma above.

Let us first prove this in the special case: for all $k \in \mathbb{Z}$

$$\eta(u) = |u-k|, \quad Q(u) = \operatorname{sgn}(u-k)\cdot(F(u)-F(k))$$

$$\lambda \cdot \left[|u_r - k| - |u_l - k| \right] \geq \left[\operatorname{sgn}(u_r - k)(F(u_r) - F(k)) \right] - \left[\operatorname{sgn}(u_l - k)(F(u_l) - F(k)) \right]$$

If $k > u_l > u_r \Rightarrow \lambda \cdot (u_l - u_r) \geq F(u_l) - F(u_r)$.

If $k < u_r < u_l \Rightarrow \lambda \cdot (u_l - u_r) \leq F(u_l) - F(u_r)$

(so this is if and only if as these inequalities imply R-H condition)

Finally, let $k \in [u_r, u_l]$. Then $k = \alpha u_r + (1-\alpha) u_l$ and

$$\lambda \left[k - u_r - (u_l - k) \right] \geq (-1)(F(u_r) - F(k)) - (F(u_l) - F(k))$$

$$\lambda \underbrace{\left[2k - u_r - u_l \right]}_{= 2\alpha u_r + 2(1-\alpha) u_l - u_r - u_l \,=\, (2\alpha - 1) u_r + (1 - 2\alpha) u_l \,=\, (1-2\alpha)(u_l - u_r)} \geq 2F(k) - F(u_r) - F(u_l)$$

$$\Rightarrow (F(u_l) - F(u_r))(1 - 2\alpha) \geq 2F(k) - F(u_r) - F(u_l)$$

$$\Rightarrow 2F(k) \leq F(u_l)(2 - 2\alpha) + F(u_r)(2\alpha)$$

and this inequality is satisfied. The general case follows by density.

THEOREM 2 Let $u_l < u_r$. Then, rarefaction wave is an entropy solution

$$u(x,t) = \begin{cases} u_l & x < F'(u_l)t \\ (F')^{-1}\left(\frac{x}{t}\right) & \text{otherwise} \\ u_r & x > F'(u_r)t \end{cases}$$

PROOF: From the lecture, we know that $u(x,t)$ satisfies eqn. pointwisely. Since solution is continuous, it satisfies Lemma above and the conclusion follows.

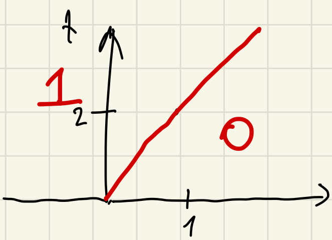
SUMMARY From uniqueness of entropy solutions to be proven next week, we can fully characterize Riemann problem for scalar conservation laws:

- $\to$ $u_l > u_r \Rightarrow$ SHOCK,
- $\to$ $u_l < u_r \Rightarrow$ RAREFACTION WAVE.

COROLLARY If u is a distributional solution satisfying equation a.e. and u is continuous $\Rightarrow$ u is an entropy solution.

EXERCISE For Burger's equation $u_t + \left(u^2/2\right)_x = 0$, we take entropy $\eta(u) = u^3$, $Q(u) = \frac{3}{4}u^4$. Let $u_0(x) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$

Then speed of shock is $\lambda = \frac{F(1)-F(0)}{1-0} = \frac{1}{2}$ (i.e. $x(t) = \frac{1}{2}t$)



$$\lambda \cdot (\eta(u_r) - \eta(u_l)) = -\frac{1}{2}$$

$$Q(u_r) - Q(u_l) = -\frac{3}{4}.$$

We know that entropy inequality is equivalent to

$$\lambda\left(\eta(u_r)-\eta(u_l)\right) \geq Q(u_r)-Q(u_l) \quad (*)$$

The example shows that the entropy is not conserved along the shock. It serves as a good method to remember $(*)$.