Hyperbolic Conservation Laws Tutorial

Topic 3: Strong continuity in time for entropy solutions

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1. Strong continuity in time.

1. Strong continuity in time

The following shows that entropy condition brings regularization-in-time effect. Our target is the following result:

<u>THEOREM</u> (strong continuity in time)

Suppose that $u(t,x)$ is a distributional solution with initial data $u_0 \in L^\infty_{loc}$

$$\int \partial_t \xi \cdot u + \int \nabla \varphi \cdot F(u) + \int \varphi(x,0)\, u_0(x) = 0. \quad \forall \varphi$$

Assume that for some entropy/entropy-flux pair (η, Q) such that η is uniformly convex, we have $\eta(u)_t + \operatorname{div} Q(u) \leq 0$

$$\int \partial_t \varphi \cdot \eta(u) + \int \nabla \varphi \cdot Q(u) + \int \varphi(x,0)\ (u_0) \geq 0 \quad \forall \varphi \geq 0.$$

Then, $t \mapsto u(t,\cdot)$ has strongly continuous representative except some countable set $\mathcal{F}$ such that $0 \notin \mathcal{F}$. $\blacksquare$

<u>REMARK</u>

The result shows in what sense the initial condition is satisfied. Namely, when $t \to 0$,

$$u(t,x) \longrightarrow u_0(x) \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

This is usually interesting for parabolic/hyperbolic PDEs where we don't control time derivative (because it does not appear in weak formulation.

Some general ideas how to prove continuity-in-time

The idea is to start with weak-* continuity, i.e. to find continuous representative for map:

$$t \mapsto \int_{\mathbb{R}^d} u(t,x)\, \psi(x)\, dx \qquad (\psi \in C_c^\infty(\mathbb{R}^d)).$$

One can write it as a limit of continuous (BV) functions:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} u^\varepsilon(t,x)\, \psi(x)\, dx$$

so we need to study sequence of functions

$$f_\psi^\varepsilon(t) = \int_{\mathbb{R}^d} u^\varepsilon(t,x)\, \psi(x)\, dx$$

This function can be studied using weak formulation.

Using some compactness results (we will discuss later why Arzela-Ascoli will not work in our case) we obtain weak-* continuous map.

The same trick will work for entropy inequality (we will transform it into equality) so $t \mapsto \eta(u)$ is also weakly-* continuous. Then, we prove that $t \mapsto u$ and $t \mapsto \eta(u)$ are weakly-* cont $\Rightarrow$ $t \mapsto u$ is strongly continuous (think about $\eta(u) = u^2$)

1.1. Characterization of nonnegative distributions

We start by rewriting distributional inequality

$$u_t + \operatorname{div} \alpha(t,x) \leq 0$$

as equality

$$u_t + \operatorname{div} \alpha(t,x) = \mu.$$

<u>LEMMA 1</u> ("$\leq$" $\Rightarrow$ "$=$")

Let T be a nonnegative distribution i.e. $T(\phi) \geq 0$ for all $\phi \in C_c^\infty$ such that $\phi \geq 0$. Then T extends to C_c and is in fact a locally bounded measure.

<u>PROOF</u> We only have to prove that T can be extended to C_c and then use Riesz representation theorem (any nonnegative functional on C_c is locally bounded measure).

Let $\psi \in C_c^\infty$. We want $|T(\psi)| \lesssim \|\psi\|_\infty$. We could use an observation that $\|\psi\|_\infty \mp \psi \geq 0$ but this is not compactly supported function. So let $R > 0$ be such that $\operatorname{supp} \psi \subset B(0,R)$ and let φ_R be usual cutoff

$$\varphi_R = \begin{cases} 1 & B(0,R) \\ \text{affine} & B(0,R+1) \setminus B(0,R) \\ 0 & \mathbb{R}^d \setminus B(0,R+1). \end{cases}$$

(smooth)

Then $\varphi_R \|\psi\|_\infty \pm \psi \ge 0$, $\varphi_R \|\psi\|_\infty \pm \psi \in C_c^\infty$ so

$$\pm T(\psi) + \|\psi\|_\infty T(\varphi_R) \ge 0 \Rightarrow |T(\psi)| \le \underbrace{T(\varphi_R)}_{\text{constant depending only on support.}} \|\psi\|_\infty$$

■

<u>COROLLARY</u> For entropy condition $\eta(u)_t + \operatorname{div} Q(u) \le 0$ we have $\eta(u)_t + \operatorname{div} Q(u) = -\mu(t,x)$

(note that this measure μ depends on η and Q!). ■

<u>EXERCISE</u> When this measure is finite?

$$|\mu|(B_x(0,R) \times B_t(0,R)) \le \int \varphi_R^x \varphi_R^t \, d\mu(t,x) =$$

$$\le \int_{\substack{R \le t \le R+1 \\ |x| \le R+1}} \underbrace{|\partial_t \varphi_R^t|}_{\le 1} \eta(u) + \int_{\substack{R \le |x| \le R+1 \\ |t| \le R+1}} \underbrace{|\nabla \varphi_R^x|}_{\le 1} Q(u) + \text{initial cond.}$$

So we need sth like $\eta(u) \in L^1_{t,x}$, $Q(u) \in L^1_{t,x}$ so we can send $R \to \infty$. ■

1.2. Upgrade: weak convergence to strong convergence.

We say that $f:\mathbb{R}\to\mathbb{R}$ is uniformly convex if for all $t\in[0,1]$

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)-t(1-t)\phi(|x-y|)$$

for some $\phi:\mathbb{R}\to\mathbb{R}$ continuous, nonnegative and vanishing only at 0.

<u>LEMMA</u> $u_n \xrightarrow{*} u$, $f(u_n)\xrightarrow{*} f(u)$ in L^∞_{loc} $\Rightarrow$

$u_n \longrightarrow u$ strongly in L^p_{loc} $1\leq p<\infty$.

<u>PROOF:</u> By uniform convexity with $x=u$, $y=u_n$, $t=\frac{1}{2}$

$$\frac{1}{4}\int\phi(|u_n-u|)\,\Psi(x)\,dx+\int f\left(\frac{u_n+u}{2}\right)\Psi(x)\,dx\leq$$

$$\leq\frac{1}{2}\int f(u_n)\,\Psi(x)\,dx+\frac{1}{2}\int f(u)\,\Psi(x)\,dx$$

where Ψ is compactly supported and nonnegative (the purpose of Ψ is to localize). Hence

$$\limsup_{n\to\infty}\frac{1}{4}\int\phi(|u_n-u|)\,\Psi(x)\,dx\leq\int f(u)\,\Psi(x)\,dx-$$

$$-\liminf\int f\left(\frac{u_n+u}{2}\right)\Psi(x)\,dx.$$

Note that $\frac{u_n+u}{2} \xrightarrow{*} u$. Since $u(x) \mapsto \int f(u(x))\,\psi(x)$ is l.s.c. on (say L^2_{loc}) we have

$$\int f(u)\psi \le \liminf \int f\left(\frac{u_n+u}{2}\right)\psi$$

$$\Rightarrow \; - \liminf_{n\to\infty} \int f\left(\frac{u_n+u}{2}\right)\psi \le - \int f(u)\psi$$

$$\Rightarrow \; \limsup_{n\to\infty} \int \phi(|u_n-u|)\psi \le 0$$

Hence, $\phi(|u_n-u|) \to 0$ in L^1_{loc} and, up to a subseq., $\phi(|u_{n_k}-u|) \to 0$ a.e. It follows that $u_{n_k} \to u$ a.e and the conclusion follows. ∎

WEAK LOWER SEMICONTINUITY ABOVE: For Banach space $(E, \|\cdot\|)$, $F: E \to \mathbb{R}$ is l.s.c iff $\forall_\lambda$ $\{u \in E : F(u) \le \lambda\}$ is closed. So if F is strongly continuous and convex

$$\{u \in E : F(u) \le \lambda\} \text{ is strongly closed and convex. } \forall_\lambda.$$

Then, Mazur lemma implies that

$$\{u \in E : F(u) \le \lambda\} \text{ is weakly closed and convex. } \forall_\lambda$$

(here I use convexity). This implies weak lower semicont.

Comment on convexity assumption:

If $f \in C^2$ and $f'' > m > 0$, we obtain from Taylor's expansion

$$f(y) = f(x) + f'(x)(y-x) + f''(z)(y-x)^2/2$$
$$\geq f(x) + f'(x)(y-x) + \frac{m}{2}(y-x)^2$$

We apply the inequality above twice with

- $x = ty_1 + (1-t)y_2$, $y = y_1 \Rightarrow y - x = (1-t)(y_1 - y_2)$

$$f(y_1) \geq f(x) + f'(x)(1-t)(y_1-y_2) + \frac{m}{2}(1-t)^2(y_1-y_2)^2 \qquad (1)$$

- $x = ty_1 + (1-t)y_2$, $y = y_2 \Rightarrow y - x = t(y_2 - y_1)$.

$$f(y_2) \geq f(x) + f'(x)t(y_2-y_1) + \frac{m}{2}t^2(y_1-y_2)^2 \qquad (2)$$

Multiply (1) with t and (2) with $(1-t)$ to deduce

$$f(ty_1 + (1-t)y_2) \leq tf(y_1) + (1-t)f(y_2)$$
$$- \frac{m}{2}\left[(1-t)^2 t + t^2(1-t)\right](y_1-y_2)^2$$

$$\Rightarrow f(ty_1 + (1-t)y_2) \leq tf(y_1) + (1-t)f(y_2)$$
$$- \frac{m}{2}t(1-t)(y_1-y_2)^2$$

so f is uniformly convex. ∎

1.3 BV functions, Helly compactness result

Ref: Bresson, Hyp. systems of conservation laws

$$BV([a,b]) = \left\{ f:[a,b]\to\mathbb{R} \;:\; \|f\|_{BV} = \sup \sum_{i=1}^{N} |f(x_{i+1})-f(x_i)| < \infty \right\}$$

and supremum above is taken over all partitions $\{x_i\}_{i=1}^N$ such that $a \le x_1 \le \ldots \le x_N \le b$.

We also define for $f\in BV[a,b]$:

$$V_f(x) = \sup \sum_{i=1}^{N} |f(x_{i+1})-f(x_i)|$$

where supremum is taken over all partitions $\{x_i\}_{i=1}^N$ such that $a \le x_1 \le \ldots \le x_N \le x$ (so this is variation up to the level x).

Some properties of BV functions:

1) $f\in BV$ can be decomposed into $f = f_1 - f_2$ where f_1, f_2 are nondecreasing (Indeed: $f = V_f - (V_f - f)$).

2) Consequence: f has countably many points of discontinuity (property of monotone functions).

3) f has well-def. left and right limits (because if $x_n \to x^-$, $\sum |f(x_{i+1})-f(x_i)| < \infty \Rightarrow |f(x_{i+1})-f(x_i)| \to 0$ and $\{f(x_i)\}$ is Cauchy in $\mathbb{R}$.

LEMMA (Helly)

Suppose that $u_n : \mathbb{R} \to \mathbb{R}^n$ such that

$$\|u_n\|_{BV} \leq C_1, \quad \|u_n\|_\infty \leq C_2.$$

Then, there is $u \in BV \cap L^\infty$ and subsequence u_{n_k} s.t.

- $u_{n_k} \longrightarrow u$ pointwisely, (*)
- $\|u\|_{BV} \leq C_1$, $\|u\|_\infty \leq C_2$.

PROOF:

STEP 1: Let $U_n(x)$ be variation of $u_n(x)$. Then,

- $0 \leq U_n(x) \leq C_1$,
- U_n is nondecreasing,
- $\forall_{p_1 \leq x \leq y \leq p_2} \quad |u_n(x) - u_n(y)| \leq U_n(p_2) - U_n(p_1)$

Only the last property seems to be nontrivial. It can be written equivalently as:

$$U_n(p_1) + |u_n(x) - u_n(y)| \leq U_n(p_2)$$

and now it is trivial.

STEP 2: We can choose subsequence such that $\{U_n\}$ converges at each $x \in \mathbb{Q}$ to some U.

This is done by a diagonal procedure.

STEP 3: Let $J_n = \left\{ x \in [a,b] : \lim_{\substack{y \in \mathbb{Q} \\ y \to x^+}} U(y) - \lim_{\substack{y \in \mathbb{Q} \\ y \to x^-}} U(y) \geq \frac{1}{n} \right\}$

Then, J_n is countable and so, $\bigcup J_n$ is also countable.

Indeed, U maps $\mathbb{Q}$ into $[0, C_1]$ and has to be nondecreasing. There can be only $C_1 \cdot n$ jumps of size $\frac{1}{n}$.

STEP 4: There is further subsequence such that

$$u_n(x) \to u(x) \text{ for ALL } x \in [a,b]$$

Indeed, we can choose it so that the convergence occurs on $\mathbb{Q} \cup \bigcup_{n \geq 1} J_n$. If $x \notin \mathbb{Q} \cup \bigcup_{n \geq 1} J_n \Rightarrow x \notin \mathbb{Q}, \ x \notin J_n \ \forall n$.

Fix n and observe that $x \notin J_n$ implies that there are $p_1 < x < p_2$ such that $U(p_2) - U(p_1) \leq \frac{2}{n}$, $p_1, p_2 \in \mathbb{Q}$.

We will prove that $\{u_n(x)\}$ is a Cauchy sequence (in $\mathbb{R}$).

$$\limsup_{k,h \to \infty} |u_k(x) - u_h(x)| \leq$$

$$\leq \limsup_{k \to \infty} |u_k(x) - u(p_1)| + \limsup_{h \to \infty} |u_h(x) - u(p_1)|$$

$$= \limsup_{k \to \infty} |u_k(x) - u_k(p_1)| + \limsup_{h \to \infty} |u_h(x) - u_h(p_1)| \leq$$

because we can write $|u_k(x) - u_k(p_1)| + |u_k(p_1) - u(p_1)|$ and we have convergence at rational numbers.

$$\leq 2 \limsup_{k\to\infty} |u_k(x) - u_k(p_1)| \leq 2 \limsup_{k\to\infty} |U_k(x) - U_k(p_1)| \leq$$

$$\leq 2 \limsup_{k\to\infty} |U_k(p_2) - U_k(p_1)| = 2\,|U(p_2) - U(p_1)| \quad \text{as } p_1, p_2 \text{ are in } \mathbb{Q}$$

$$\leq \frac{4}{n}.$$ As n is arbitrary, $\{u_n(x)\}$ converges.

STEP 5: The limit u satisfies $\|u\|_\infty \leq C_2$, $\|u\|_{BV} \leq C_1$.

The first point is clear. For the second, we use pointwise convergence,

$$\sum_{i=1}^{N} |u(x_{i+1}) - u(x_i)| = \lim_{k\to\infty} \sum_{i=1}^{N} |u_k(x_{i+1}) - u_k(x_i)| \leq$$

$$\leq \sup_k \|u_k\|_{BV} \leq C_1$$

$$\Rightarrow \|u\|_{BV} \leq C_1.$$ ∎

1.4 Weak-* continuity for $u_t + \operatorname{div} \alpha = \mu$.

(Ref: Dafermos, Lemma 1.3.3)

LEMMA Let $u \in L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ with $u_0 \in L^\infty_{loc}(\mathbb{R}^d)$, $\alpha \in L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$, μ locally bounded measure such that

$$u_t + \operatorname{div} \alpha(t,x) = \mu, \quad u(0,x) = u_0(x).$$

Then, the map $t \mapsto u(t,\cdot)$ is weakly-* continuous. (i.e. it has weakly-* continuous representative) except some countable set $\mathcal{F}$ (and $0 \notin \mathcal{F}$!).

In particular, for this representative,

$$w^*\text{-}\lim_{t \to 0} u(t,\cdot) = u_0(\cdot). \quad \text{(locally)}$$

This means $\forall_{\varphi \in L^1_{loc}} \int u(t,x)\varphi(x)\,dx \to \int u_0(x)\varphi(x)\,dx$

when $t \to 0$.

PROOF: Fix $T > 0$, $\tau > 0$, $R > 0$. We write weak formulation

$$\iint \varphi_t u + \iint \nabla\varphi \cdot \alpha + \iint \varphi \, d\mu + \int \varphi(x,0)\, u_0(x) = 0.$$

Let φ be supported on $(\tau, T-\tau) \times B(0,R) \ni (t,x)$. Test equation with $\varphi(t,x)\,\eta^\varepsilon(x)\,\eta^\varepsilon(t)$, supported for $(0,T) \times B(0,R)$ where $\varepsilon < \tau$ (so that mollification is welldefined). Pointwisely, we have:

$$u_t^{\varepsilon} = \operatorname{div} \alpha^{\varepsilon}(t,x) + \mu^{\varepsilon}(t,x) \qquad (t,x) \in (\tau, T-\tau) \times B(0,R).$$

Clearly, $(t,x) \mapsto u^{\varepsilon}(t,x)$ is very smooth and satisfies equality above pointwisely. We claim that sequence

$$f_{\psi}^{\varepsilon}(t) = \int_{\mathbb{R}^d} u^{\varepsilon}(t,x)\, \psi(x)\, dx \qquad (\varepsilon < \tau,\ \psi \in C_c^{\infty}(\mathbb{R}^d))$$

satisfies assumptions of Helly Lemma. Indeed, multiply equation with $\psi(x)$ and integrate from some s to some t. It follows that

$$f_{\psi}^{\varepsilon}(t) - f_{\psi}^{\varepsilon}(s) = \int_s^t \int_{\mathbb{R}^d} \nabla \psi(x)\, \alpha^{\varepsilon}(t,x) + \int_s^t \int_{\mathbb{R}^d} \psi(x)\, d\mu^{\varepsilon}(t,x)$$

$$\Rightarrow |f_{\psi}^{\varepsilon}(t_1) - f_{\psi}^{\varepsilon}(t_2)| \leq \|\nabla \psi\|_{\infty} \int_{t_1}^{t_2} \int_{\operatorname{supp}\psi} |\alpha^{\varepsilon}(t,x)|\, dt\, dx$$

$$+ \|\psi\|_{\infty}\, \mu^{\varepsilon}([t_1,t_2] \times \operatorname{supp}\psi).$$

Note that μ^{ε} is a function (or measure with density),

Given any partition $\tau < t_1 < t_2 < \ldots < t_n < T-\tau$ we have

$$\sum_{i=1}^{n-1} |f_{\psi}^{\varepsilon}(t_{i+1}) - f_{\psi}^{\varepsilon}(t_i)| \leq \|\nabla \psi\|_{\infty} \int_{t_1}^{t_n} \int_{\operatorname{supp}\psi} |\alpha^{\varepsilon}(t,x)|\, dt\, dx \quad \Rightarrow$$

$$+ \|\psi\|_{\infty}\, \mu^{\varepsilon}([t_1,t_n] \times \operatorname{supp}\psi).$$

$$\sum_{i=1}^{n-1} \left| f_\psi^\varepsilon(t_{i+1}) - f_\psi^\varepsilon(t_i) \right| \le \|\nabla \psi\|_\infty \int_\tau^{T-\tau} \int_{\operatorname{supp}\psi} |\alpha^\varepsilon(t,x)| \, dt\, dx \; +$$

$$+ \; \|\psi\|_\infty \, \mu^\varepsilon\big([\tau, T-\tau] \times \operatorname{supp}\psi\big)$$

By standard properties of mollifiers,

$$\int_\tau^{T-\tau} \int_{\operatorname{supp}\psi} |\alpha^\varepsilon(t,x)| \, dt\, dx \le \int_0^T \int_{(\operatorname{supp}\psi)^\varepsilon} |\alpha(t,x)| \, dt\, dx \qquad \alpha \in L^1_{loc,x} L^1_{loc,t}$$

bound indep. of τ !

For measure term we write

$$\int_{[\tau, T-\tau]\times \operatorname{supp}\psi} \int_{\mathbb{R}\times\mathbb{R}^d} \eta_\varepsilon(t-s)\,\eta_\varepsilon(x-y)\, d\mu(s,y)\, dt\, dx$$

$$\mu^\varepsilon\big([\tau, T-\tau] \times \operatorname{supp}\psi\big) \le \mu\big([\tau, T-\tau] \times \operatorname{supp}\psi\big) \le$$

$$\le \mu\big([0,T] \times \operatorname{supp}\psi\big).$$

Using Helly's Lemma, we can choose a subsequence conv. everywhere to a BV function on $(\tau, T-\tau)$. Applying diagonal argument with $\tau = \frac{1}{n}$, we get a $BV(0,T)$ version of

$$t \mapsto \int u(t,x)\, \psi(x)\, dx =: f_\psi$$

However, its set of discontinuity points depends on ψ. To overcome this issue, we choose a countable dense subset of $C_c^\infty(B(R,0))$, say $\{\psi_k\}_{k\in\mathbb{N}}$ and apply Helly Lemma diagonally

This yields a countable set $\mathcal{F}$ s.t. for each φ_k

$$[0,T]\setminus\mathcal{F} \ni t \mapsto \int u(t,x)\varphi_k(x)\,dx$$

We claim that the same set $\mathcal{F}$ works for all $\varphi \in C_c(\mathbb{R}^d)$.

Indeed, for arbitrarily φ, we find φ_k s.t. $\|\varphi-\varphi_k\|_\infty \leq \varepsilon$.

If t is a point of continuity

$$\left|\int u(t,x)\varphi(x)\,dx - \int u(s,x)\varphi(x)\,dx\right| \leq$$

$$\leq \left|\int u(t,x)\varphi_k(x)\,dx - \int u(t,x)\varphi(x)\,dx\right| +$$

$$+ \left|\int u(s,x)\varphi_k(x)\,dx - \int u(s,x)\varphi(x)\,dx\right|$$

$u \in L^\infty_{loc,t} L^1_{loc,x}$

$$+ \left|\int u(s,x)\varphi_k(x)\,dx - \int u(t,x)\varphi_k(x)\,dx\right|$$

$$\leq 2\|u\|\,\varepsilon + \left|\int u(s,x)\varphi_k(x)\,dx - \int u(t,x)\varphi_k(x)\,dx\right|$$

$$\Rightarrow \limsup_{s\to t} \left|\int u(s,x)\varphi(x)\,dx - \int u(t,x)\varphi(x)\,dx\right| \leq 2\|u\|\,\varepsilon.$$

Similar argument shows that the same set of continuity points works for weaker classes like $\varphi \in L^1(\mathbb{R}^d)$ — this however depends on integrability of u so that $\int u(s,x)\varphi(x) < \infty$.

We can also use diagonal argument for parameters R and T.

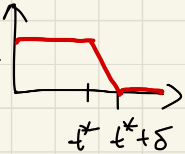
Finally, we need to check that:

(1) $0 \notin \mathcal{F}$ i.e. f_φ is continuous at 0.

(2) $f_\varphi(0) = \int u_0(x)\varphi(x)\,dx.$

This follows from weak formulation. Let $t^* \in \mathcal{F}$. Then,

$$\iint \varphi_t u + \iint \nabla\varphi \cdot \alpha + \iint \varphi \, d\mu + \int \varphi(x,0)\, u_0(x) = 0$$

with $\Psi(x)\varphi^\delta(t)$ where $\varphi^\delta(t) =$ 

. This yields:

$$-\frac{1}{\delta}\int_{t^*}^{t^*+\delta}\int_{\mathbb{R}^d} u(t,x)\Psi(x) + \iint \nabla\Psi(x)\varphi^\delta(t)\cdot\alpha$$

$$+ \iint \Psi(x)\varphi^\delta(t)\, d\mu(t,x) + \int \Psi(x) u_0(x)\, dx = 0.$$

As t^* is continuity point, we send $\delta \to 0$ and obtain

$$-\int_{\mathbb{R}^d} u(t^*,x)\Psi(x)\,dx + \int_0^{t^*}\int_{\mathbb{R}^d} \nabla\Psi(x)\,\alpha(t,x)$$

$$+ \int_0^{t^*}\int_{\mathbb{R}^d} \Psi(x)\, d\mu(t,x) + \int \Psi(x) u_0(x)\, dx = 0$$

We send $t^* \to 0$ which finally proves

$$\lim_{t^*\to 0} \int_{\mathbb{R}^d} u(t^*,x)\Psi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\Psi(x)\,dx.$$

$\blacksquare$

PROOF OF THE MAIN RESULT: Applying lemma to equations

$$u_t + \operatorname{div} F(u) = 0, \quad \eta(u)_t + \operatorname{div} Q(u) = -\mu$$

where η is uniformly convex (as we work with scalar conserv. laws we can choose $\eta(u) = u^2$).

It follows that $t \mapsto u(t,\cdot)$, $t \mapsto \eta(u(t,\cdot))$ are weakly-* continuous. Lemma from Section 1.2 shows that $t \mapsto u(t,\cdot)$ is strongly continuous.

MORE COMMENTS ON CONTINUITY:

1. We proved that bounded entropy solutions are continuous as the maps

$$[0,T] \setminus \mathcal{F} \ni t \mapsto u(t,\cdot) \in L^1_{loc}(\mathbb{R}^d)$$

and $\mathcal{F} \neq 0$. In fact, so-called "fine structure of BV solutions" to conservation laws shows that $\mathcal{F}$ is empty.

2. For Kruzkhov technique (proof of uniqueness), it is suff. that:

$$\lim_{t \to 0^+} u(t,\cdot) = u_0(\cdot) \quad \text{in } L^1_{loc}(\mathbb{R}^d) \text{ (strongly!)}$$

3. If $\mu = 0$, one can apply different compactness argument in Lemma above to get that $\mathcal{F} = \emptyset$. $\Rightarrow$ HOMEWORK.