Hyperbolic Conservation Laws Tutorial

Topic 4: Continuity estimates for vanishing viscosity method. Existence of entropy solutions.

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1. Main result.

Theorem Let u be the solution of

$$u_t(t,x) + \operatorname{div} f(u(t,x)) = \varepsilon \Delta u(t,x) \;, \quad \varepsilon > 0. \qquad (P)$$
$$u(0,x) = u_0(x)$$

where $u_0 \in L^1(\mathbb{R}^n; [a,b])$ (in part: $u_0 \in L^\infty$) and

$$\int_{\mathbb{R}^n} |u_0(x+y) - u_0(x)| \, dx \leq \omega(|y|) \qquad \forall_{y \in \mathbb{R}^n}$$

for some modulus of continuity ω. Then, there is a constant $C = C(a,b)$ s.t.

- $$\int_{\mathbb{R}^n} |u(t,x+y) - u(t,x)| \, dx \leq \omega(|y|) \qquad \forall_{y \in \mathbb{R}^n}$$

- $$\int_{\mathbb{R}^n} |u(t+h,x) - u(t,x)| \, dx \leq$$
$$\leq C\left[h^{2/3} + \varepsilon h^{1/3}\right] \|u_0\|_{L^1} + 2\,\omega(h^{1/3})$$

<u>Comment</u> on the assumption with modulus of continuity.

Any L^1 function satisfies that with

$$\omega(h) = \sup_{|y| \leq h} \int_{\mathbb{R}^n} |u(x+y) - u(x)| \, dy$$

This function is clearly monotone when $h \to 0$.

Moreover, $\omega(h) \to 0$ when $h \to 0$. This is clear when u is $C_c(\mathbb{R}^n)$ because u is uniformly continuous. For general $u \in L^1(\mathbb{R}^n)$, fix $\varepsilon > 0$ and take u_n s.t. $\|u_n - u\|_1 \leq \varepsilon$.

Then

$$\int_{\mathbb{R}^n} |u(x+y) - u(x)| \, dy \leq 2 \int_{\mathbb{R}^n} |u(x) - u_n(x)| \, dx$$
$$+ \int_{\mathbb{R}^n} |u_n(x+y) - u_n(x)| \, dx$$

Hence, $\omega(h) \leq 2\varepsilon + \sup_{|y| \leq h} \int_{\mathbb{R}^n} |u_n(x+y) - u_n(x)| \, dy$

$\Rightarrow \limsup_{h \to 0} \omega(h) \leq 2\varepsilon$. $\square$

2. Continuity in space

We want to prove an estimate

$$\int_{\mathbb{R}^n} |u(t,x+y) - u(t,x)|\,dx \leq \omega(|y|).$$

Recall contraction property for (P). When u, v solve (P) we have

$$\int_{\mathbb{R}^n} |u(t,x) - v(t,x)|\,dx \leq \int_{\mathbb{R}^n} |u(s,x) - v(s,x)|\,dx.$$

$$\leq \int_{\mathbb{R}^n} |u(0,x) - v(0,x)|\,dx$$

Note that $u(t,x+y)$ solves (P) with initial condition $u_0(x+y)$. Hence contraction property implies

$$\int_{\mathbb{R}^n} |u(t,x+y) - u(t,x)|\,dx \leq \int_{\mathbb{R}^n} |u_0(x+y) - u_0(x)|\,dx \leq$$

$$\leq \omega(|y|) \quad \text{by assumption.}$$

(Btw: using contraction property with $v=0$ we get

$$\int_{\mathbb{R}^n} |u(t,x)|\,dx \leq \int_{\mathbb{R}^n} |u_0(x)|\,dx \Rightarrow u \in L^\infty_t L^1_x \quad).$$

3. Continuity in time

We want to control $\int_{\mathbb{R}^n} |u(t+h,x) - u(t,x)|\, dx$ by some modulus of continuity. Fix some $h>0$. As equation is satisfied pointwisely (for $t>0$), we can multiply by $\varphi(x)$ and integrate in time

$$u_t + \operatorname{div} f(u(t,x)) = \varepsilon \Delta u$$

$$\int_{\mathbb{R}^n} \varphi(x)\big(u(t+h,x) - u(t,x)\big) = \int_{\mathbb{R}^n}\int_t^{t+h} \varepsilon\, \Delta\varphi(x) u(s,x)\, ds\, dx$$

$$+ \int_{\mathbb{R}^n}\int_t^{t+h} f(u(s,x))\, \nabla\varphi(x)\, ds\, dx$$

$\rightsquigarrow$ t fixed here!

Wlog, $f(0)=0$. Let $v(x) = u(t+h,x) - u(t,x)$. To conclude the proof, we would like to take $\varphi(x) = \operatorname{sgn} v(x)$ but:

- φ is not compactly supported: this is not a problem. In fact, formulation above can be generalized for bounded and smooth φ because $u \in L^\infty_t L^1_x$. Moreover, as $u \in L^\infty$ and f is locally Lipschitz

$$|f(u(t,x))| = |f(u(t,x)) - f(0)| \lesssim C|u(t,x)| \in L^\infty_t L^1_x.$$

- such Ψ is not regular enough. To this end, we introduce regularization with standard mollifier $\eta_\delta : \mathbb{R}^n \to \mathbb{R}$ and δ will depend on h in the future.

So we take $\Psi_\delta(x) = \eta_\delta * \operatorname{sgn}(v(x))$. Recall that

- $\nabla \Psi_\delta(x) = \nabla \eta_\delta * \operatorname{sgn}(v)$
- $\Delta \Psi_\delta(x) = \Delta \eta_\delta * \operatorname{sgn}(v)$

By Young's inequality,

$$\| f * g \|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

we have estimates for derivatives

$$\| \nabla \Psi_\delta \|_\infty \leq \| \nabla \eta_\delta \|_1 \overbrace{\| \operatorname{sgn} v \|_\infty}^{=1} \leq \int_{\mathbb{R}^n} \frac{1}{\delta^n} (\nabla \eta)\left(\frac{x}{\delta}\right) \frac{1}{\delta} dx$$

$$= \int_{B_1(\delta)} \frac{1}{\delta^{n+1}} |\nabla \eta| \left(\frac{x}{\delta}\right) = \int_{B_1(0)} \frac{1}{\delta} |\nabla \eta|(y) \, dy \leq \frac{C_1(\eta)}{\delta}$$

Similarly, $\quad \| \Delta \Psi_\delta \|_\infty \leq \dfrac{C_2(\eta)}{\delta^2}.$

We come back to the integral identity

$$\int_{\mathbb{R}^n} \psi^\delta(x)\, v(x)\, dx = \int_{\mathbb{R}^n}\int_t^{t+h} \varepsilon\, \Delta\psi^\delta(x)\, u(s,x)\, ds\, dx + \int_{\mathbb{R}^n}\int_t^{t+h} f(u(s,x))\, \nabla\psi^\delta(x)\, ds\, dx .$$

The terms on the (RHS) are easily estimated:

- $$\left| \int_{\mathbb{R}^n}\int_t^{t+h} \varepsilon\, \Delta\psi^\delta(x)\, u(s,x)\, ds\, dx \right| \leq$$
$$\leq \frac{\varepsilon\, C_2(\eta)}{\delta^2} \cdot h \sup_{s\in[t,t+h]} \int_{\mathbb{R}^n} |u(s,x)|\, dx \leq \frac{\varepsilon\, C_2(\eta)\, h}{\delta^2} \int_{\mathbb{R}^n} |u_0(x)|\, dx$$

- $$\left| \int_{\mathbb{R}^n}\int_t^{t+h} f(u(s,x))\, \nabla\psi^\delta(x)\, ds\, dx \right| \leq$$ f is locally Lip, u is bounded
$$\leq \frac{C_1(\eta)\, h}{\delta} \sup_{s\in[t,t+h]} \int_{\mathbb{R}^n} |u(s,x)|\, dx \leq \frac{C_1(\eta)\, h}{\delta} \int_{\mathbb{R}^n} |u_0(x)|\, dx$$

We see we need to choose $\delta = h^{1/3}$. Then, these terms are controlled with:

$$\left[C_1(\eta)\, h^{2/3} + \varepsilon C_2(\eta)\, h^{1/3}\right].$$

Finally, we need to replace $\int_{\mathbb{R}^n} \psi^\delta(x)\, v(x)\, dx$ with $\int_{\mathbb{R}^n} |v(x)|\, dx$. Observe that

1. $$\int_{\mathbb{R}^n} \psi^\delta(x)\, v(x)\, dx = \int_{\mathbb{R}^n} (\operatorname{sgn} v * \eta_\delta)(x)\, v(x)\, dx =$$
$$= \int_{\mathbb{R}^n} \operatorname{sgn} v(x) \;\; v * \eta_\delta(x)\, dx.$$

2. $|v(x)| = \operatorname{sgn}(v(x)) \cdot v(x).$

Therefore,

$$\int_{\mathbb{R}^n} \left[\psi^\delta(x) v(x) - |v(x)|\right] dx = \int_{\mathbb{R}^n} \operatorname{sgn} v(x) \cdot \left[v * \eta_\delta(x) - v(x)\right] dx$$

$$= \int_{\mathbb{R}^n} \operatorname{sgn} v(x) \int_{\mathbb{R}^n} \eta_\delta(y) \left[v(x-y) - v(x)\right] dy\, dx.$$

It follows that

$$\left| \int_{\mathbb{R}^n} \left(\psi^\delta(x) v(x) - |v(x)|\right) dx \right| \leq \int_{\mathbb{R}^n} \eta_\delta(y) \int_{\mathbb{R}^n} \left| v(x-y) - v(x) \right| dx\, dy$$

$|x-y| \leq \delta$

$$\left| \int_{\mathbb{R}^n} \left(\psi^\delta(x) v(x) - |v(x)| \right) dx \right| \leq \int_{\mathbb{R}^n} \underset{\substack{\uparrow \\ |y| \leq \delta}}{\eta_\delta(y)} \int_{\mathbb{R}^n} | v(x-y) - v(x) | \, dx \, dy$$

Recall that $v(x) = u(t+h, x) - u(t,x)$. From continuity in space estimates the latter term can be estimated with

$$\int_{\mathbb{R}^n} \eta_\delta(y) \underbrace{\int_{\mathbb{R}^n} | u(t+h, x-y) - u(t+h, x) | \, dx}_{\leq \omega(|y|) \leq \omega(\delta) = \omega(h^{1/3})} \, dy \; +$$

$$+ \int_{\mathbb{R}^n} \eta_\delta(y) \underbrace{\int_{\mathbb{R}^n} | u(t, x-y) - u(t,x) | \, dx}_{\leq \omega(|y|) \leq \omega(\delta) = \omega(h^{1/3})} \, dy$$

$$\leq 2\omega(h^{1/3}).$$

We conclude

$$\int_{\mathbb{R}^n} |v(x)| \, dx \leq C_1(\eta) h^{2/3} + \varepsilon C_2(\eta) h^{1/3} + 2\omega(h^{1/3}),$$

as desired.

4. Existence of entropy solutions.

<u>Theorem</u> Let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^n)$ and f be locally Lipschitz continuous. Then there exists an entropy solution to

$$\partial_t u + \operatorname{div} f(u) = 0, \quad u(0,x) = u_0(x).$$

Moreover,

$$\int_{\mathbb{R}^n} |u(t,x)|\,dt \leq \int_{\mathbb{R}^n} |u_0(x)|\,dx, \quad \|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty.$$

<u>Proof</u>: We multiply regularization with $\eta'(u^\varepsilon)$

$$\partial_t u^\varepsilon + \operatorname{div} f(u^\varepsilon) = \varepsilon \Delta u^\varepsilon$$

to get

$$\partial_t \eta(u^\varepsilon) + \operatorname{div} Q(u^\varepsilon) = (\varepsilon \Delta u^\varepsilon)\, \eta'(u^\varepsilon)$$

where (η, Q) is an admissible entropy / entropy-flux pair. Moreover

$$(\varepsilon \Delta u^\varepsilon)\, \eta'(u^\varepsilon) = \varepsilon \Delta \eta(u^\varepsilon) - \underbrace{\varepsilon |\nabla u^\varepsilon|^2 \eta''(u^\varepsilon)}_{\geq 0}$$

$$\leq \varepsilon \Delta \eta(u^\varepsilon)$$

so we discover

$$\partial_t \eta(u^\varepsilon) + \operatorname{div} Q(u^\varepsilon) \leq \varepsilon \Delta \eta(u^\varepsilon)$$

Take a smooth test function and rewrite it in the sense of distributions

$$-\int \eta(u^\varepsilon)\partial_t \varphi - \int Q(u^\varepsilon)\cdot \nabla\varphi - \int \eta(u_0)\varphi(x,0) \leq \qquad (\ast)$$
$$\leq \int \varepsilon \Delta\varphi\, \eta(u^\varepsilon)$$

We need strong convergence of $\{u^\varepsilon\}$ to pass to the limit. As η, Q are continuous, we only need to get $u^\varepsilon \to u$ a.e. and invoke Dominated Convergence ($|\eta(u^\varepsilon)|, |Q(u^\varepsilon)| \leq C$ and φ has compact support).

We want to use Kolmogorov theorem on each $[0,R]\times B_R(0)$ ball in time + space. Indeed,

- $\{u^\varepsilon\}$ is bounded in $L^1([0,R]\times B_R(0))$;

$$\int_0^R \int_{B_R(0)} |u^\varepsilon(s,x)|\, dx\, ds \leq \int_0^R \int_{\mathbb{R}^d} |u_0(x)|\, dx\, ds$$
$$\leq R \int_{\mathbb{R}^d} |u_0(x)|\, dx < \infty.$$

- translations are uniformly continuous:

$$\int_0^R \int_{\mathbb{R}^n} |u^\varepsilon(t,x+y) - u^\varepsilon(t,x)| \, dx \, dt \leq \tilde{\omega}(|y|)$$

$$\int_0^R \int_{\mathbb{R}^n} |u^\varepsilon(t+h,x) - u^\varepsilon(t,x)| \, dx \, dt \leq \tilde{\omega}(|h|).$$

This is satisfied due to our continuity estimates. Hence $\forall_R$ we can choose subsequence converging in $L^1([0,R] \times B_R(0))$. We can choose further subsequence converging a.e. and so by diagonal argument we can choose subsequence $u^{\varepsilon_{n_k}} \to u$ a.e. in $\mathbb{R} \times \mathbb{R}^n$.

Passing to the limit in $(*)$

$$-\int \eta(u)\,\partial_t \varphi - \int Q(u) \cdot \nabla \varphi - \int \eta(u_0)\,\varphi(0,x)\,dx \leq 0$$

as $\varepsilon \int \Delta \varphi \, \eta(u^\varepsilon) \to 0$. It follows that u satisfies

$$\partial_t \eta(u) + \operatorname{div} Q(u) \leq 0 \quad \text{in the sense of distributions.}$$

so that u is an entropy solution (recall that we know that entropy inequality implies $\partial_t u + \operatorname{div} f(u) = 0$).

Finally, $\|u\|_\infty \leq \|u_0\|_\infty$ follows from $\|u^\varepsilon\|_\infty \leq \|u_0\|_\infty$ and pointwise convergence while L^1 estimates follows from Fatou lemma:

$$\int_{\mathbb{R}^n} |u(t,x)|\, dx \leq \int_{\mathbb{R}^n} \liminf_{\varepsilon \to 0} |u^\varepsilon(t,x)|\, dx \leq$$

$$\leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} |u^\varepsilon(t,x)|\, dx \leq \int_{\mathbb{R}^n} |u_0(x)|\, dx.$$

$\square$.