Hyperbolic Conservation Laws Tutorial

Topic 5: Comments on Kruzkhov doubling variables method.

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[We work here with solutions s.t. $u \in C([0,T]; L^1_{loc})$, $u_0 \in L^\infty$].

Recall that to prove uniqueness of entropy solutions we take two solutions $u(t,x)$, $v(\tilde{t},\tilde{x})$. Both satisfy entropy inequality

$$\partial_t \eta(u(t,x)) + \operatorname{div} Q(u(t,x)) \leq 0.$$

We use Kruzkhov entropies: $\eta(u,v) = |u-v|$ with flux $Q(u,v) = \operatorname{sgn}(u-v)(f(u)-f(v))$. Then

$$\partial_t \eta(u(t,x), v(\tilde{t},\tilde{x})) + \operatorname{div}_x Q(u(t,x), v(\tilde{t},\tilde{x})) \leq 0$$

$$\partial_{\tilde{t}} \eta(u(t,x), v(\tilde{t},\tilde{x})) + \operatorname{div}_{\tilde{x}} Q(u(t,x), v(\tilde{t},\tilde{x})) \leq 0$$

We test both of them with $\phi(t,x,\tilde{t},\tilde{x}) \geq 0$.

$$(1) \quad \int_0^\infty \int_{\mathbb{R}^d} \left[\phi_t \, \eta(u,v) + \nabla_x \phi \, Q(u,v)\right] dt\,dx \; +$$

$$+ \int_{\mathbb{R}^d} \phi(0,x,\tilde{t},\tilde{x}) \, \eta(u_0,v) \, dx \geq 0$$

(2) $\int_0^\infty \int_{\mathbb{R}^d} \left[\phi_{\tilde{t}}\, \eta(u,v) + \nabla_{\tilde{x}} \phi \cdot Q(u,v) \right] d\tilde{x}\, d\tilde{t}$

$$+ \int_{\mathbb{R}^d} \phi(t,x,0,\tilde{x})\, \eta(u, v_0)\, d\tilde{x} \geq 0.$$

Integrate both in x and t. Add together to get

$$\iiiint (\partial_t + \partial_{\tilde{t}}) \phi \; \eta(u,v) + (\nabla_x + \nabla_{\tilde{x}}) \phi \cdot Q(u,v)\, dx\, d\tilde{x}\, dt\, d\tilde{t}$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(0,x,\tilde{t},\tilde{x})\, \eta(u_0(x), v(\tilde{t},\tilde{x}))\, dx\, d\tilde{x}\, d\tilde{t}$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t,x,0,\tilde{x})\, \eta(u(\tilde{t},\tilde{x}), v_0(x))\, dx\, d\tilde{x}\, dt \geq 0$$

Now, we need to clean doubled variables. Consider

$$\phi(t,x,\tilde{t},\tilde{x}) = \Psi\left(\frac{x+\tilde{x}}{2}, \frac{t+\tilde{t}}{2}\right) \xi_\varepsilon\left(\frac{x-\tilde{x}}{2}\right) \xi_\varepsilon\left(\frac{t-\tilde{t}}{2}\right)$$

and we expect in the limit (when $\varepsilon \to 0$) $t = \tilde{t}$, $x = \tilde{x}$.

Moreover, if Ψ is supported in the time arg. for $\frac{t+\tilde{t}}{2} \geq \varepsilon_0$ and $\varepsilon < \varepsilon_0/4$ we have $\begin{cases} \frac{t+\tilde{t}}{2} \geq \varepsilon_0 \\ |t-\tilde{t}| < 2\varepsilon < \varepsilon_0/2 \end{cases}$. Hence, when

When $t=0$, there are no $\tilde{t}$ s.t. $\phi(0,x,\tilde{t},\tilde{x}) \neq 0$. Similarly, for $\tilde{t}=0$. Hence, pink terms in the integral identity above can be removed.

$$\iiiint (\partial_t+\partial_{\tilde{t}})\left[\psi\left(\frac{x+\tilde{x}}{2},\frac{t+\tilde{t}}{2}\right)\zeta_\varepsilon\left(\frac{t-\tilde{t}}{2}\right)\zeta_\varepsilon\left(\frac{x-\tilde{x}}{2}\right)\right]\eta(u(t,x),v(\tilde{t},\tilde{x}))$$

$$+\iiiint (\nabla_x+\nabla_{\tilde{x}})\left[\psi\left(\frac{x+\tilde{x}}{2},\frac{t+\tilde{t}}{2}\right)\zeta_\varepsilon\left(\frac{t-\tilde{t}}{2}\right)\zeta_\varepsilon\left(\frac{x-\tilde{x}}{2}\right)\right]Q(u(t,x),v(\tilde{t},\tilde{x}))$$

$$\geq 0.$$

TROUBLES: Differentiation of mollifier.

We want to send $\varepsilon \to 0$. We introduce new variables

$$\begin{cases} y=\dfrac{x+\tilde{x}}{2} \\ \tilde{y}=\dfrac{x-\tilde{x}}{2} \end{cases} \quad \begin{cases} s=\dfrac{t+\tilde{t}}{2} \\ \tilde{s}=\dfrac{t-\tilde{t}}{2} \end{cases} \Rightarrow \begin{cases} x=y+\tilde{y} \\ \tilde{x}=y-\tilde{y} \\ t=s+\tilde{s} \\ \tilde{t}=s-\tilde{s} \end{cases}$$

Term with time derivatives:

$$(\partial_t+\partial_{\tilde{t}})\left[\psi\left(\frac{x+\tilde{x}}{2},\frac{t+\tilde{t}}{2}\right)\zeta_\varepsilon\left(\frac{t-\tilde{t}}{2}\right)\right]=$$

$$=2\cdot\psi_t\left(\frac{x+\tilde{x}}{2},\frac{t+\tilde{t}}{2}\right)\cdot\frac{1}{2}\cdot\zeta_\varepsilon\left(\frac{t-\tilde{t}}{2}\right)+\psi\left(\frac{x+\tilde{x}}{2},\frac{t+\tilde{t}}{2}\right)\left[\zeta_\varepsilon'-\zeta_\varepsilon'\right]$$

$$=\psi_t\left(\frac{x+\tilde{x}}{2},\frac{t+\tilde{t}}{2}\right)\zeta_\varepsilon\left(\frac{t-\tilde{t}}{2}\right).$$

After changing variables (I don't need to worry about jacobian as it is constant and we consider inequality ≥ 0).

$$\iiiint \psi_t(y,s)\, \zeta_\varepsilon(y')\, \zeta_\varepsilon(s')\, \eta\big(u(s+\tilde{s}, y+\tilde{y}),\ v(s-\tilde{s}, y-\tilde{y})\big)$$

LEMMA 1. $\zeta_\varepsilon \xrightarrow{*} \delta_0$ as $\varepsilon \to 0$ in $\mathcal{M}$.

PROOF. We need to check that for all $\varphi \in C_0$

$$\int \varphi(x)\zeta_\varepsilon(x)\,dx \longrightarrow \int \varphi(x)\,d\delta_0(x) = \varphi(0)$$

This is always the same: $\left(\int \zeta_\varepsilon = 1\right)$

$$\left|\int \varphi(x)\zeta_\varepsilon(x) - \varphi(0)\right| = \left|\int (\varphi(x)-\varphi(0))\,\zeta_\varepsilon(x)\,dx\right| \leq$$

$$\leq \int |\varphi(x)-\varphi(0)|\,\zeta_\varepsilon(x)\,dx$$ and use uniform cont.

$\square$

LEMMA 2. If $f \in L^1_{loc}$, ω – modulus of continuity and ψ smooth compactly supported

$$\iint \psi(x) f(x+\omega(y))\,\eta_\varepsilon(y)\,dy\,dx \to \int \psi(x) f(x)\,dx$$

PROOF: $\left| \iint \psi(x)\,[f(x+\omega(y)) - f(x)]\,\eta_\varepsilon(y)\,dy\,dx \right|$

difference $|(x+\omega(y))-x| \leq \omega(\varepsilon)$.

so this converges to 0 when f is assumed to be in C_c.

For general f, as always, fix $\gamma > 0$ and find f_γ s.t.

$$\| f_\gamma - f \|_{L^1(\mathrm{supp}\,\psi)^1} \leq \gamma .$$

hull with diameter 1.
($\varepsilon < 1$)

As always:

$$\left| \iint \psi(x) \left(f(x+\omega(y)) - f(x) \right) \eta_\varepsilon(y)\, dx\, dy \right| \leq$$

$$\leq \left| \iint \psi(x) \left(f(x+\omega(y)) - f_\gamma(x+\omega(y)) \right) \eta_\varepsilon(y)\, dx\, dy \right|$$
$$+ \left| \iint \psi(x) \left(f(x) - f_\gamma(x) \right) \eta_\varepsilon(y)\, dx\, dy \right|$$
$$+ \left| \iint \psi(x) \left(f_\gamma(x+\omega(y)) - f_\gamma(x) \right) \eta_\varepsilon(y)\, dx\, dy \right|$$
$$\leq 2\gamma \|\psi\|_\infty + \left| \iint \psi(x) \left(f_\gamma(x+\omega(y)) - f_\gamma(x) \right) \eta_\varepsilon(y)\, dx\, dy \right|$$

Hence,

$$\limsup_{\varepsilon \to 0} \left| \iint \psi(x) \left(f(x+\omega(y)) - f(x) \right) \eta_\varepsilon(y)\, dx\, dy \right| \leq$$
$$\leq 2\gamma \|\psi\|_\infty \to 0 \quad \text{as } \gamma \to 0. \quad \square$$

LEMMA 3. If $f \in L^1_{loc}$, $\omega_1, \dots, \omega_n$ some modulus of continuity and ψ compactly supported

$$\iint \psi(x) f(x_1 \pm \omega_1(y), x_2 \pm \omega_2(y), \dots, x_n \pm \omega_n(y))\, \eta_\varepsilon(y)\, dy\, dx$$
$$\longrightarrow \int \psi(x) f(x_1, x_2, \dots, x_n)\, dx$$

PROOF: If f is C_c, the difference of arguments in

$$|f(x_1 \pm \omega_1(y), x_2 \pm \omega_2(y), \dots, x_n \pm \omega_n(y)) - f(x_1, \dots, x_n)|$$

is controlled by $\sup_{1 \le i \le n} \omega_i(\varepsilon)$. We conclude as above. $\square$

Coming back to the term with time derivative.

$$\iiiint \varphi_t(y,s)\, \xi_\varsigma(y')\, \xi_\varepsilon(s')\, \eta(u(s+\tilde{s}, y+\tilde{y}), v(s-\tilde{s}, y-\tilde{y}))$$

Note that η is Lipschitz continuous so $\eta(u(\dots), v(\dots))$ is L^1_{loc} and we can use convergence lemma above for $(\tilde{s}, \tilde{y})$ to get in the limit:

$$\iint \varphi_t(y,s)\, \eta(u(s,y), v(s,y))\, dy\, ds .$$

Term with spatial gradient:

$$\iiiint (\nabla_x + \nabla_{\tilde{x}}) \left[\varphi\left(\frac{x+\tilde{x}}{2}, \frac{t+\tilde{t}}{2}\right) \xi_\varepsilon\left(\frac{t-\tilde{t}}{2}\right) \xi_\varsigma\left(\frac{x-\tilde{x}}{2}\right) \right] Q(u(t,x), v(\tilde{t}, \tilde{x}))$$

We compute gradient:

$$(\nabla_x + \nabla_{\tilde{x}}) \left[\varphi\left(\frac{x+\tilde{x}}{2}, \frac{t+\tilde{t}}{2}\right) \xi_\varsigma\left(\frac{x-\tilde{x}}{2}\right) \right] =$$

$$= 2 (\nabla_x \varphi)\left(\frac{x+\tilde{x}}{2}, \frac{t+\tilde{t}}{2}\right) \frac{1}{2}\, \xi_\varsigma\left(\frac{x-\tilde{x}}{2}\right) + 0 =$$

$$= (\nabla_x \psi)\left(\frac{x+\tilde{x}}{2}, \frac{t+\tilde{t}}{2}\right) \zeta_\varepsilon\left(\frac{x-\tilde{x}}{2}\right).$$

Therefore, this term reads

$$\iiiint (\nabla_x \psi)\left(\frac{x+\tilde{x}}{2}, \frac{t+\tilde{t}}{2}\right) \zeta_\varepsilon\left(\frac{x-\tilde{x}}{2}\right) \zeta_\varepsilon\left(\frac{t-\tilde{t}}{2}\right) Q\left(u(t,x), v(\tilde{t},\tilde{x})\right)$$

Again, we change variables and we don't care about Jacobian:

$$\iiiint (\nabla_x \psi)(y,s)\, \zeta_\varepsilon(\tilde{y})\, \zeta_\varepsilon(\tilde{s})\, Q\left(u(s+\tilde{s}, y+\tilde{y}), v(s-\tilde{s}, y-\tilde{y})\right)$$

and we use Lemma above to get in the limit $\varepsilon \to 0$:

$$\iint (\nabla_x \psi)(y,s)\, Q\left(u(s,y), v(s,y)\right)$$

Hence,

$$\iint \psi_t(t,x)\, \eta\left(u(t,x), v(t,x)\right) dt\, dx \; +$$

$$+ \iint \nabla \psi(t,x)\, Q\left(u(t,x), v(t,x)\right) dt\, dx \geq 0$$

$\blacksquare$