Hyperbolic Conservation Laws Tutorial

Topic 6: Toolbox for compensated compactness.

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1. Negative Sobolev spaces and connection with $-\Delta$ operator.
2. div-curl lemma
3. Murat lemma

Motivation: formulate general results on products on weakly converging sequences (say $v_n \rightharpoonup v$, $w_n \rightharpoonup v$; when $v_n \cdot w_n \to v \cdot w$?). In general, it's not true (consider $\sin^2(nx)$).

1. Negative Sobolev spaces and connection with $-\Delta$ operator.

Definition: we write $W^{-1,q}(U)$ for the dual space of $W_0^{1,q'}(U)$ equipped with the usual dual norm

$$\|\varphi\| = \sup_{\substack{u \in W_0^{1,q'}(U) \\ \|u\| \le 1}} \varphi(u)$$

We write $\varphi \in W^{-1,q}_{loc}(U)$ if $\varphi \in W^{-1,q}(K)$ for each compact $K \subset\subset U$.

Remark: Some people define $W^{-1,q}(U)$ as a dual space of $W_0^{1,q}(U)$. Unless $q=2$, it is always important to set up notation.

Remark: Mostly, we will be concerned with $W^{-1,q}_{loc}(\mathbb{R}^n)$ so we restrict our attention to $W^{-1,q}(\Omega)$ for Ω bdd.

Remark: We set up hierarchy of negative Sobolev spaces. For usual Sobolev spaces we have $W_0^{1,q_1} \subset W_0^{1,q_2}$ when $q_2 < q_1$. Here,

$$W^{-1,q_1}(\Omega) \subset W^{-1,q_2}(\Omega) \qquad q_2 < q_1$$

Indeed, when $\varphi \in W^{-1,q_1}(\Omega) \Rightarrow \varphi$ is bdd functional on $W_0^{1,q_1'}(\Omega) \supset W_0^{1,q_2'}(\Omega) \Rightarrow \varphi \in W_0^{-1,q_2}(\Omega)$. $\square$.
(as $q_1' > q_2'$)

There is a nice trick to move between positive and negative Sobolev spaces.

FACT 1. Let $1 < q < \infty$. Then, for each $\varphi \in W^{-1,q}(\Omega)$ there exists a unique solution to

$$\begin{aligned} -\Delta u_\varphi &= \varphi \qquad && \Omega \\ u &= 0 && \partial\Omega \end{aligned}$$

(in the sense that $\int \nabla u_\varphi \cdot \nabla v = \varphi(v) \quad \forall v \in W_0^{1,q'}(\Omega)$).

Moreover, $-\Delta$ is bijective as a linear operator

$$-\Delta : W_0^{1,q} \longrightarrow W^{-1,q}$$

and there are constants $\frac{1}{C}\|\varphi\|_{W^{-1,q}} \leq \|u_\varphi\|_{W_0^{1,q}} \leq$

$\leq C\|\varphi\|_{W^{-1,q}}$.

PROOF ($q=2$). Recall Lax-Milgram lemma: given coercive, bounded and bilinear form $a(u,v)$ on Hilbert space $\mathcal{H}$, for each $\ell \in \mathcal{H}^*$ $\exists!_{u \in \mathcal{H}}$ s.t. $a(u,v) = \ell(v)\ \forall_{v \in \mathcal{H}}$.

Let $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$, $\mathcal{H} = W_0^{1,2}(\Omega)$, $\ell = \varphi \in \mathcal{H}^* = W_0^{-1,2}(\Omega)$. Hence, we get the unique solution u_φ. Moreover

$$\|u_\varphi\|^2_{W_0^{1,2}} = \int_\Omega |\nabla u_\varphi|^2 = \varphi(u_\varphi) \leq \|\varphi\| \, \|u_\varphi\|_{W_0^{1,2}}$$

$$\Rightarrow \|u_\varphi\|_{W_0^{1,2}} \leq \|\varphi\|.$$

Let $(-\Delta)^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$ so that $\varphi \xmapsto{(-\Delta)^{-1}} u_\varphi$.

- the map is injective: indeed when $u_\varphi = 0$ $\Rightarrow \varphi = 0$.
- the map is surjective: if $u_\varphi \in W_0^{1,2}$ is fixed, we may define $\varphi = -\Delta u_\varphi \in W_0^{-1,2}$ defined as
$$\varphi(\phi) = \int \nabla u_\varphi \cdot \nabla \phi$$
It follows that $(-\Delta)^{-1}(\varphi) = u_\varphi$.

It follows that $(-\Delta)^{-1}$ is a bounded isomorphism between two Banach spaces. Inverse Mapping Theorem implies that its inverse is also bounded. $\square$.

PROOF (case $q \in (1,\infty)$). This involves few techniques from singular integrals, Calderon-Zygmund theory and regularity theory for elliptic equations. The complete proof can be found here:

https://people.math.ethz.ch/~salamon/PREPRINTS/pde.pdf

(lecture notes "PDEs in geometry" by D. Salamon @ ETH Zurich $\Rightarrow$ Chapter 3 and 4 contain the result). $\square$

In case $q=2$, we will need better regularity estimate:

FACT 2 (H^2 regularity). Let $u \in H^1_0(\Omega)$ be a weak sln to $-\Delta u = f$ with $f \in L^2(\Omega)$ and Dirichlet bdd cond. Then $u \in H^2$ and there exists C s.t. $\|u\|_{H^2} \leq C\|f\|_2$.

Proof: Theorem 4 is Section 6.3, Evans. This is based on testing equation with difference quotients. $\square$.

Using Facts 1 and 2, we can prove everything else.

LEMMA (Interpolation in negative Sobolev spaces).

Let $1 < q_0 < q < q_1 < \infty$ with $\frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_0}$. Suppose that $\varphi \in W^{-1,q_0}(\Omega) \cap W^{-1,q_1}(\Omega)$. Then, $\varphi \in W^{-1,q}$ and we have estimate

$$\|\varphi\|_{W^{-1,q}} \leq C(q,q_0,q_1) \, \|\varphi\|_{W^{-1,q_0}}^{1-\lambda} \, \|\varphi\|_{W^{-1,q_1}}^{\lambda}.$$

PROOF: We transform the problem to positive Sobolev spaces and apply Hölder inequality.

Indeed, $\varphi \in W^{-1,q_0} \Rightarrow \exists_{u_\varphi \in W^{1,q_0}} \quad \varphi = -\Delta u_\varphi$

$\varphi \in W^{-1,q_1} \Rightarrow \exists_{\tilde{u}_\varphi \in W^{1,q_1}} \quad \varphi = -\Delta \tilde{u}_\varphi$

$\Downarrow$

$\varphi \in W^{-1,q} \Rightarrow \exists_{\tilde{\tilde{u}}_\varphi \in W^{1,q}} \quad \varphi = -\Delta \tilde{\tilde{u}}_\varphi$

We claim that $u_\varphi = \tilde{u}_\varphi = \tilde{\tilde{u}}_\varphi$. Indeed, we have uniqueness for each of these problems. For instance, if $\tilde{\tilde{u}}_\varphi \in W^{1,q} \Rightarrow \tilde{\tilde{u}}_\varphi \in W^{1,q_0}$ so it solves eq. also in W^{1,q_0}.

Applying Hölder inequality

$$\|u_\varphi\|_q \leq \|u_\varphi\|_{q_0}^{1-\lambda} \|u_\varphi\|_{q_1}^{\lambda}.$$

Coming back to negative Sobolev spaces:

$$\|\varphi\|_{W^{-1,q}} \leq C \|\varphi\|_{W^{-1,q_0}}^{1-\lambda} \|\varphi\|_{W^{-1,q_1}}^{\lambda}. \qquad \square.$$

2. div-curl Lemma.

LEMMA. Let $\{v_n\}$, $\{w_n\}$ be two vector fields such that:

- they are bounded in $L^2_{loc}(\mathbb{R}^n)$
- $\{\operatorname{div} v_n\}$ is compact in $H^{-1}_{loc}(\mathbb{R}^n)$
- $\{\operatorname{curl} w_n\}$ is compact in $H^{-1}_{loc}(\mathbb{R}^n; M^{n\times n})$.

Suppose that $v_n \rightharpoonup v$, $w_n \rightharpoonup w$ in L^2_{loc}. Then we have $w_n \cdot v_n \to w \cdot v$ in $\cancel{L^1_{loc}}$. in the sense of distr.

PROOF. The main tool is to apply laplacian trick.
We fix some bdd $\Omega \subset \mathbb{R}^n$ and define u_n as

$$\begin{cases} -\Delta u_n = w_n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \Rightarrow \|u_n\|_{H^2} \leq C.$$

As a motivation, we compute (i is fixed):

$$\operatorname{div}\left(u^i_{n,x_j} - u^j_{n,x_i}\right) = \sum_{j=1}^n \left(u^i_{n,x_j} - u^j_{n,x_i}\right)_{x_j} =$$

$$= \sum_{j=1}^n u^i_{n,x_jx_j} - \sum_{j=1}^n u^j_{n,x_ix_j} = -w^i_n - \operatorname{div}(u_{n,x_i})$$

$$\underbrace{\operatorname{div}\left(u^i_{n,x_j} - u^j_{n,x_i}\right)}_{-y^i_n} = -w^i_n - \underbrace{\operatorname{div}\left(u_{n,x_i}\right)}_{z^i_n}$$

$\Rightarrow \quad w^i_n = z^i_n + y^i_n \quad$ or $\quad w_n = z_n + y_n$ (as vectors).

Sequence z_n can be written as $-\nabla \operatorname{div} u_n$.

Sequence y_n is compact strongly in $L^2(\Omega)$: indeed, $\operatorname{curl} w_n$ is compact in H^{-1} $\Rightarrow$ $\operatorname{curl} u_n$ is compact in H^1 so $\operatorname{div} \operatorname{curl} u_n$ is compact in $L^2(\Omega)$.

Now, we write $\int v_n \cdot w_n \varphi = \int v_n (z_n + y_n) \varphi =$
$= \int v_n \cdot z_n \varphi + \int v_n \cdot y_n \varphi$.

For the second term, $v_n \rightharpoonup v$ in L^2 but $y_n \to y$ in L^2 so $v_n y_n \rightharpoonup v y$ in L^1. For the first term

$$\int v_n \cdot z_n \cdot \varphi = -\int v_n \cdot \nabla \operatorname{div} u_n \varphi = \int \operatorname{div} v_n \cdot \operatorname{div} u_n \varphi$$

$$+ \int v_n \cdot \nabla \varphi \operatorname{div} u_n$$

Warning: the 2nd term has to be understood as:

$(\mathrm{div}\, v_n)(\mathrm{div}\, u_n\, \varphi)$ as $\mathrm{div}\, v_n$ is only in H^{-1}.

Term $(\mathrm{div}\, v_n)(\mathrm{div}\, u_n\, \varphi)$: we know that $\mathrm{div}\, v_n \to \mathrm{div}\, v$ in H^{-1} and $\mathrm{div}\, u_n\, \varphi \longrightarrow \mathrm{div}\, u\, \varphi$ (as u_n is bdd in H^2 $\Rightarrow$ $\mathrm{div}\, u_n$ bdd in H^1 $\Rightarrow$ $\mathrm{div}\, u_n$ compact in L^2).

Hence

$$\left|(\mathrm{div}\, v_n)(\mathrm{div}\, u_n\, \varphi) - (\mathrm{div}\, v)(\mathrm{div}\, u\, \varphi)\right| \leq$$

$$\leq \underbrace{(\mathrm{div}\, v_n - \mathrm{div}\, v)}_{\to 0 \text{ in } H^{-1}}\underbrace{(\mathrm{div}\, u_n\, \varphi)}_{\text{bdd in } H^1} + \mathrm{div}\, v\, \underbrace{(\mathrm{div}\, u_n - \mathrm{div}\, u)}_{\to 0 \text{ in } H^1.}\varphi$$

Term $\int v_n \cdot \nabla \varphi\, \mathrm{div}\, u_n$: as u_n bdd in H^2 $\Rightarrow$ $\mathrm{div}\, u_n$ is compact in L^2 (strongly) by Rellich. As $v_n \rightharpoonup v$ weakly, product of weakly and strongly converging sequences, converges to the appropriate limit. We conclude

$$\int v_n z_n \varphi \longrightarrow (\mathrm{div}\, v)(\mathrm{div}\, u\, \varphi) + \int v\, \nabla \varphi\, \mathrm{div}\, u$$

$$= -\int v\, (\nabla\, \mathrm{div}\, u)\, \varphi = \int v\, z\, \varphi.$$

Finally, $\int v_n w_n \varphi \longrightarrow \int v z \varphi + \int v y \varphi =$

$$= \int v(z+y)\varphi = \int v w \varphi. \qquad \square.$$

[in homeworks: one simple application and another result of compensated compactness type

compact in time × compact in space = compact.].

3. Murat Lemma and some compact subsets in negative Sobolev spaces.

For applications in conservation laws, we need a better understanding what is compact in negative Sobolev spaces.

Example: if $\{f_k\}$ is compact in $L^2(\Omega)$, then $\partial_x f_k$ is compact in $H^{-1}(\Omega)$. Indeed, when $f_k \to f$ in $L^2(\Omega)$,

$$\|\partial_x f_k - \partial_x f\|_{H^{-1}} = \sup_{\substack{\varphi \in H_0^1(\Omega) \\ \|\varphi\| \leq 1}} \int (f_k - f)\,\partial_x \varphi \leq$$

$$\leq \|f_k - f\|_2 \, \|\partial_x \varphi\|_2 \leq \|f_k - f\|_2 \longrightarrow 0. \qquad \square$$

Example 2: in view of $-\Delta u_k = f_k$ we have $\{u_k\}$ is bdd in $W_0^{1,q}(\Omega) \Leftrightarrow \{f_k\}$ is bdd in $W^{-1,q}(\Omega)$.

$\square$

Now, we prove that bounded measures form a compact set in $W^{-1,q}$. This requires that $W_0^{1,q'}(\Omega) \Subset C(\overline{\Omega})$, which holds when q is sufficiently large ($q' > n$, n is the dimension of the space).

LEMMA. Let $q > n$. Let $\{\mu_n\}$ be bounded in $\mathcal{M}(\Omega)$ with total variation. Then $\{\mu_n\}$ is compact in $W^{-1,q}(\Omega)$.

PROOF. Let $B = \{u \in W_0^{1,q'}(\Omega) : \|u\| \leq 1\}$. $B \subset C(\bar{\Omega})$ and actually $B \subset\subset C(\bar{\Omega})$.

Fix $\varepsilon > 0$ and choose $N(\varepsilon)$-functions $\varphi_1, \dots, \varphi_{N(\varepsilon)} \in C(\bar{\Omega})$ such that

$$B \subset \bigcup_{i=1}^{N(\varepsilon)} B(\varphi_i, \varepsilon)$$

We may always find subsequence $\mu_{n_k} \to \mu$ weakly-*. We claim that $\mu_{n_k} \to \mu$ in $W^{-1,q}$. Indeed,

$$\|\mu_{n_k} - \mu\|_{W^{-1,q}} = \sup_{\varphi \in B} \int \varphi \, d(\mu_{n_k} - \mu) =$$

$$= \sup_{\varphi \in B} \left[\int (\varphi - \varphi_{i,\varphi}) \, d(\mu_{n_k} - \mu) + \int \varphi_{i,\varphi} \, d(\mu_{n_k} - \mu) \right]$$

$$\leq 2\varepsilon \sup_n \|\mu_n\|_{TV} + \sup_{1 \leq i \leq N(\varepsilon)} \int \varphi_{i,\varphi} \, d(\mu_{n_k} - \mu)$$

$$\leq 2\varepsilon \sup_n \|\mu_n\|_{TV} \quad \text{as } n_k \to \infty.$$

As $\varepsilon > 0$ is arbitrary, we conclude the proof $\square$.

$q' > n$ means $\frac{1}{q'} < \frac{1}{n} \iff \frac{1}{q} > 1 - \frac{1}{n} = \frac{n-1}{n} \iff$

$\iff q < \frac{n}{n-1} = (1)^*$.

[Homework] If $\{f_n\}$ bdd in $L^p(\Omega)$ then $\{f_n\}_{n\in\mathbb{N}}$ is compact in $W^{-1,q}$ for $q < p^*$.

LEMMA (Murat). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, A compact in $W^{-1,2}(\Omega)$, B bounded in $\mathcal{M}(\Omega)$ and C bounded in $W^{-1,p}(\Omega)$ for some $p > 2$. Then, any D s.t. $D \subset (A+B) \cap C$ is compact in $W^{-1,2}(\Omega)$.

PROOF: Note that B is compact in $W^{-1,q}$ for $q < \frac{n}{n-1} = 2$. It follows that $A+B$ is compact in $W^{-1,q}$ for any $q < 2$. Finally, D is compact in $W^{-1,q}$ $(q<2)$ and bounded in $W^{-1,p}$ $(p>2)$. Interpolation inequality proves that D is compact in any $W^{-1,r}$ $(r < p)$, in particular $W^{-1,2}$ as desired. $\square$.

(Various generalizations of this lemma are possible).