# Hyperbolic Conservation Laws Tutorial

## Topic 7: Introduction to kinetic functions

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1. Definition and motivation.
2. Variational characterization.
3. Connection with strong convergence and Young measures.

# 1. Definition and motivation

If $u^\varepsilon \overset{*}{\rightharpoonup} u$ in $L^\infty$, we have for all smooth $G$

$$G(u^\varepsilon) \overset{*}{\rightharpoonup} \int G(\lambda)\, d\mu_{t,x}(\lambda)$$

where $\{\mu_{t,x}\}_{t,x}$ is a Young measure generated by the sequence $\{u^\varepsilon\}_{\varepsilon>0}$. The target is to define a similar object that can encode all weak limits of nonlinearities but is a function rather than measure.

Fix $(t,x)$ s.t. $u^\varepsilon(t,x) \geq 0$. Then

$$G(u^\varepsilon) = G(0) + \int_0^{u^\varepsilon(t,x)} G'(\xi)\, d\xi =$$

$$= G(0) + \int_{\mathbb{R}} \underbrace{\mathbb{1}_{0<\xi<u^\varepsilon(t,x)}(\xi)}_{:= \chi(\xi,\, u^\varepsilon(t,x))} G'(\xi)\, d\xi$$

Similarly, when $u^\varepsilon(t,x) < 0$. We define

$$\chi(\xi,u) = \begin{cases} 1 & 0<\xi<u, \\ -1 & u<\xi<0, \\ 0 & \text{otherwise}. \end{cases}$$

**Fact** (i) $\int_{\mathbb{R}} S'(\xi)\,\chi(\xi,u)\,d\xi = S(u) - S(0)$.

(ii) $\int_{\mathbb{R}} |\chi(\xi,u) - \chi(\xi,v)|\,d\xi = |u-v|$

**Proof** (i) follows by the discussion above. For (ii) we should distinguish few cases

- $u > 0 > v \Rightarrow \chi(\xi,u) = \mathbb{1}_{0<\xi<u}, \ \chi(\xi,v) = -\mathbb{1}_{v<\xi<0}$
- $u < 0 < v \Rightarrow$ similar as above
- $u > v > 0 \Rightarrow \chi(\xi,u) - \chi(\xi,v) = \mathbb{1}_{v \leq \xi < u}$
- $u < v < 0 \Rightarrow$ similar as above. $\square$

**Corollary** If $u(x) \in L^1$, it may be tricky to define $\chi(\xi, u(x))$ as $\chi$ is discontinuous. But if $u_n \to u$ in $L^1$ and $u_n$ smooth, we see that $\chi(\xi, u_n(x))$ is a Cauchy sequence in $L^1$ converging a.e. to $\chi(\xi, u(x))$. Hence, it is well-defined. $\square$

## 2. Variational characterization.

We assume that $f$ is a given function s.t.

$$f \in L^1(\mathbb{R}), \quad f(\xi)\,\mathrm{sgn}(\xi) = |f| \leq 1.$$

**THEOREM 1**

**(i)** Let $u = \int f(\xi)\,d\xi$. Then, there is a unique fcn $m \in C_0(\mathbb{R})$ s.t. $\chi(\xi,u) - f(\xi) = \partial_\xi m(\xi)$. It is nonnegative and $\|m\|_\infty \leqslant \|f\|_1$.

**(ii)** Let $S$ be convex and Lipschitz continuous. Fix $u \in \mathbb{R}$ and consider problem

$$\inf\left\{ \int_{\mathbb{R}} S'(\xi)\, f(\xi)\,d\xi : f \in L^1,\ f\,\mathrm{sgn}\,\xi = |f| \leq 1,\ \int f = u \right\}.$$

Then, the infimum above equals $S(u) - S(0)$ and is attained for $f = \chi(\xi, u)$.

**(iii)** For $S$ unif. convex (i.e. $\exists_m$ $S(u) - mu^2$ is convex, the minimizer is unique).

Remark: Condition $u = \int f(\xi)\,d\xi$ in (i) follows from integrating equation in space.

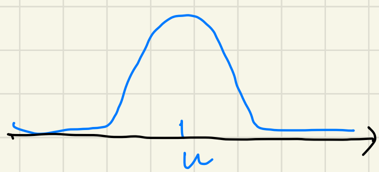
**Proof:** For (i) we set $m(\xi) = \int_{-\infty}^{\xi} \chi(\eta,u) - f(\eta)\, d\eta$.

Clearly, $m$ is continuous (even AC), $\lim_{\xi \to -\infty} m(\xi) = 0$.

Moreover, $\lim_{\xi\to\infty} m(\xi) = \int_{\mathbb{R}} \chi(\eta,u) - f(\eta)\, d\eta = 0$ as $u = \int f(\eta)\, d\eta$.

**$L^\infty$ bound:** $\chi(\eta,u) = 1\!\!1_{0<\eta<u} - 1\!\!1_{u<\eta<0}$ and $u = \int f(\eta)\, d\eta$ is fixed. Assume $u \geq 0$. Then $\chi(\eta,u) = 1\!\!1_{0<\eta<u}$. For $\eta < u$ $\underbrace{\chi(\eta,u)}_{=1} - \underbrace{f(\eta)}_{\in[-1,1]} \geq 0$ so that $m$ is nondecreasing. Then, $m$ is decreasing as $f$ is nonnegative. It follows that

$$\|m\|_\infty = m(u) = u - \int_{-\infty}^{u} f(\eta)\, d\eta = \int_{-\infty}^{\infty} - \int_{-\infty}^{u} f(\eta)\, d\eta$$
$$\leq \int_{\mathbb{R}} |f(\eta)|\, d\eta = \|f\|_1.$$




Similar story when $u \leq 0$.

**Nonnegativity:** This follows from monotonicity above and limits at infinity.

For (ii) we write $\int S'(\xi) f(\xi)\, d\xi =$

$$= \int S'(\xi) \left[ \chi(\xi,u) - m_f'(\xi) \right] d\xi =$$

$$= S(u) - S(0) - \underbrace{\int S'(\xi)\, m_f'(\xi)\, d\xi}_{\text{we need that this is nonpositive}}.$$

If $S$ is $C^2$, convexity implies $\int S'(\xi)\, m_f'(\xi)\, d\xi =$

$$= - \int S''(\xi)\, m_f(\xi)\, d\xi \leq 0 \quad \text{as } S'' \geq 0.$$

Otherwise consider $S^\varepsilon := S * \eta^\varepsilon$. It is again convex since

$$S*\eta^\varepsilon(tx+(1-t)y) = \int S(\underbrace{tx+(1-t)y-z}_{tz+(1-t)y})\, \eta^\varepsilon(z)$$

$$\leq t S*\eta^\varepsilon(x) + (1-t)\, S*\eta^\varepsilon(y)$$

It follows that $\int (S^\varepsilon)'(\xi)\, m_f'(\xi) \leq 0$. We send $\varepsilon \to 0$ using that $S^\varepsilon$ is Lipschitz (so that $S^\varepsilon$ exists a.e.).

For (iii) if we assume $S(\xi) - \alpha \xi^2$ is convex for some $\alpha > 0$ we have

$$\left(S^\varepsilon(\xi) - (\alpha \xi^2)^\varepsilon\right)'' \geq 0 \quad \text{i.e.} \quad (S^\varepsilon)'' - \left((\alpha \xi^2)^\varepsilon\right)'' \geq 0$$

$$\text{i.e.} \quad (S^\varepsilon)'' - 2\alpha \geq 0 \Rightarrow (S^\varepsilon)'' \geq 2\alpha$$

Hence, in the computation above, if $f$ is minimizer,

$$\int (S^\varepsilon)'(\xi) f(\xi)\, d\xi = S^\varepsilon(u) - S^\varepsilon(0) + \int (S^\varepsilon)'' m_f(\xi)\, d\xi$$

$$\geq S^\varepsilon(u) - S^\varepsilon(0) + 2\alpha \int m_f(\xi)\, d\xi$$

$$\xrightarrow{\varepsilon \to 0} \quad \int S'(\xi) f(\xi)\, d\xi \geq S(u) - S(0) + 2\alpha \underbrace{\int m_f(\xi)\, d\xi}_{\geq 0}$$

so $\quad 0 \geq 2\alpha \int m_f(\xi)\, d\xi \Rightarrow m_f(\xi) = 0 \ \forall \xi \Rightarrow$

$f(\xi) = \chi(\xi, u)$. $\square$.

**Remark:** We used here weaker notion of strong convexity which allows for $S$ which are not twice differentiable. If $S \in C^2$ then $S'' \geq m > 0$ so that we recover previous definition.

# 3. Connection with strong convergence and YM.

We now aim at version of fundamental theorem of YM formulated in language of kinetic fcns.

We formulate results very generally, for $u_n \rightharpoonup u$ only in $L^1_{loc}$. In view of Dunford–Pettis Thm this is equivalent to equiintegrability of $\{u_n\}$, namely

(A): $\displaystyle \omega_R(k) := \sup_{n \in \mathbb{N}} \int_{|u_n| \geq k \cap B_R} |u_n| \longrightarrow 0$

as $k \to \infty$

(B): for all $R > 0$ there is $\underbrace{\varphi_R}_{\varphi_R \geq 0}$ nondecreasing s.t. $\frac{\varphi(|x|)}{|x|} \to \infty$ when $|x| \to \infty$ and $\varphi_R(|u_n|)$ is uniformly bounded in $L^1(B_R)$.

We will use (A).

**THEOREM 2** Suppose that $u_n \to u$ in $L^1_{loc}(\mathbb{R}^d)$ and $\chi(\xi, u_n(x)) \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}^d \times \mathbb{R})$. Then,

**(A)** $f \in L^1(B \times \mathbb{R})$ for all $B \subset \mathbb{R}^d$ (ball)

**(B)** $f(\xi) \cdot \operatorname{sgn} \xi = |f(x,\xi)| \leq 1$ a.e.

**(C)** there is a family of probability measures $\{\nu_x\}_x$ such that the following identity holds in the sense of distributions

$$\frac{\partial}{\partial \xi} f(x,\xi) = \delta_0(\xi) - \nu_x(\xi)$$

**Proof:** **For (A)** we note that if $\chi(\xi, u_n(x)) \overset{*}{\rightharpoonup} f$, (so in duality with $L^1$) we take test function

$$\eta = \mathbb{1}_{\xi \in [-\tilde{R}, \tilde{R}]} \mathbb{1}_{B(R)}(x) \operatorname{sgn} f$$

$$\int f \eta = \int_{|\xi| \leq \tilde{R},\, x \in B(R)} |f| = \lim_{n \to \infty} \int_{|\xi| \leq \tilde{R},\, x \in B(R)} \chi(\xi, u_n(x)) \operatorname{sgn} f$$

$$\leq \liminf_{n \to \infty} \iint_{x \in B(R)} |\chi(\xi, u_n(x))| \, d\xi \, dx = \liminf_{n \to \infty} \int_{x \in B(R)} |u_n(x)| \, dx$$

Send $\tilde{R} \to \infty$ to get the result.

**For (B)** we first prove sign property. This follows from weak-$*$ convergence. Indeed, for $\xi > 0$ we take arbitrary $\varphi(x) \in C_c$, $\psi(\xi)$ supp. on $\xi > 0$. Then
$(\varphi \geq 0)$ $(\psi \geq 0)$

$$0 \leq \iint \varphi(x)\,\psi(\xi)\,\chi(\xi, u_n(x))\,d\xi\,dx$$

$$\longrightarrow \iint \varphi(x)\,\psi(\xi)\,f(x,\xi)\,d\xi\,dx.$$

Similarly, $f(x,\xi) \leq 0$ a.e. $x$, a.e. $\xi \leq 0$.

To see that $|f| \leq 1$ we show that $f - 1 \leq 0$ and $f + 1 \geq 0$ using method above.

**For (C)** First, we compute $\frac{\partial}{\partial \xi} \chi(\xi, u_n(x))$.

- $u_n(x) > 0$ $\quad \frac{\partial}{\partial \xi} \chi(\xi, u_n(x)) = \delta_0(\xi) - \delta_{u_n(x)}(\xi)$
- $u_n(x) < 0$ $\quad \frac{\partial}{\partial \xi} \chi(\xi, u_n(x)) = -\delta_{u_n(x)}(\xi) + \delta_0(\xi).$

Hence, we have $\frac{\partial}{\partial \xi} \chi(\xi, u_n(x)) = \delta_0 - \nu^n_x$.

Passing to the limit in the sense of distributions

$$\frac{\partial}{\partial \xi} f(x,\xi) = \delta_0 - \nu_x.$$

We only need to prove that $\{\nu_x\}$ has mass 1 (this is the same as for Young measures ... in fact $\nu_x$ is the Young measure).

First, for any smooth, compactly supp. $\psi$ we have

$$\int \psi(x)\, d\nu^n_x \longrightarrow \int \psi(x)\, d\nu_x$$

Taking sequence $\psi_k \nearrow 1$ we get

$$\nu_x(\mathbb{R}) = \int 1 \, d\nu_x = \sup_{k \in \mathbb{N}} \int \psi_k(\xi)\, d\nu_x(\xi) =$$

$$= \sup_{k \in \mathbb{N}} \lim_{n \to \infty} \int \psi_k(\xi)\, d\nu^n_x(\xi)$$

$$\leq \liminf_{n \to \infty} \sup_{k \in \mathbb{N}} \int \psi_k(\xi)\, d\nu^n_x(\xi) = 1.$$

To get lower bound we need to show that the mass does not escape to infinity.

## (decay estimate)

There is $M_R$ nondecreasing s.t. $\frac{M_R(r)}{r} \to \infty$ as $r \to \infty$.
and $\int_{B_R} M_R(u_n)$ is uniformly bounded.

$$\{|u_n|>k \cap B_R\} =$$

$$= \int_{|u_n|>k} 1 \, dx \leq \int \frac{M_R(u_n)}{M_R(k)} \, dx \leq \frac{C(R)}{M_R(k)}$$

Let $\varphi_k$ be s.t. $\varphi_k(\xi)$ supp. on $[-k,k]$, $\varphi_k = 1$ on $[-(k-1), k+1]$ and $0 \leq \varphi_k \leq 1$

$$\int_{B_R} \nu_x([-k,k]) dx = \int_{B_R}\int_{\mathbb{R}} 1\!\!1_{[-k,k]}(\xi) \, d\nu_x(\xi) \, dx$$

$$\geq \int_{B_R}\int_{\mathbb{R}} \varphi_k(\xi) \, d\nu_x(\xi) \, dx \geq \lim_{n\to\infty} \int_{B_R} \varphi_k(u_n(x)) \, dx$$

$$\geq \limsup_{n\to\infty} \int_{B_R} 1 - 1\!\!1_{|u_n| \geq (k-1)} \, dx$$

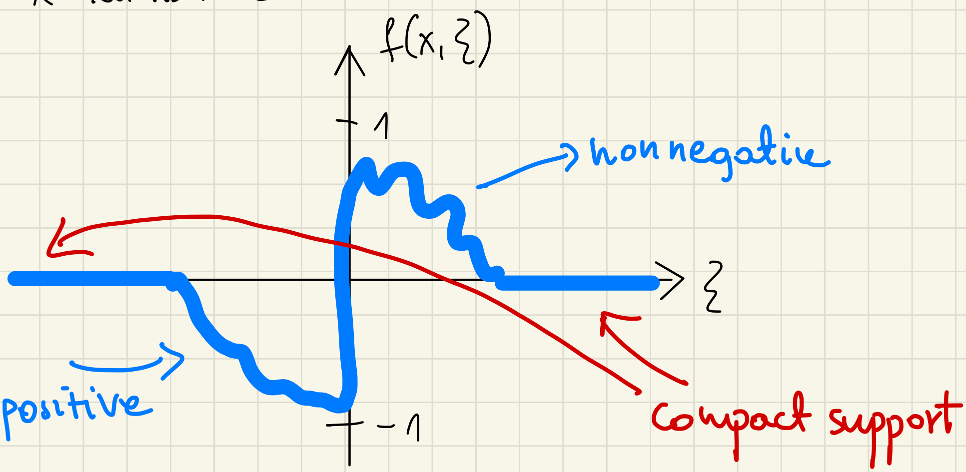
$$\geq \int_{B_R} 1\,dx - \liminf_{n\to\infty} \underbrace{\left|\{|u_n| \geq (k-1)\} \cap B_R\right|}_{\leq \frac{C_R}{M_R(k-1)}}$$

$$\geq \int_{B_R} 1\,dx - \frac{C_R}{M_R(k-1)}$$

Send $k \to \infty$ to get $\displaystyle\int_{B_R} \left[ \int d\nu_x(\xi) - 1 \right] \geq 0.$

It follows that $\displaystyle\int d\nu_x(\xi) = 1.$ ∎

Hence, we know that weak-$*$ limit of $\chi(\xi, u_n(x))$ for fixed $x$ looks like




We now prove that $f(x,\xi)$ encodes all weak limits of $S(u_n)$ for all $S$, exactly like Young measure.

**THEOREM 3.** Let $\{u_n\}_{n\in\mathbb{N}}$ be weakly compact in $L^1_{loc}(\mathbb{R}^d)$. If $u_n \rightharpoonup u$ we have

**(A)** $u(x) = \int_{\mathbb{R}} f(x,\xi)\, d\xi$

**(B)** $\int_{\mathbb{R}} |f(x,\xi)|\, d\xi =$ weak limit of $|u_n(x)|$

**(C)** More generally, if $S$ is convex, $S' \in L^\infty$ and $S(0) = 0$

$$\int S'(\xi)\, f(x,\xi)\, d\xi = \text{weak limit of } S(u_n).$$

**(D)** $u_n \to u$ strongly in $L^1_{loc}$ iff for a.e. $x$ $f(x,\xi) = \chi(\xi, u(x))$.

**Remark:** If $\{u_n\}$ is weakly compact, there is $M_R$ s.t. $\int_{B_R} M_R(|u_n|)$ is bounded. We want to find such a function for $S(u_n)$. Note that $|S(u_n)| \leq \|S'\|_\infty |u_n|$

We can take $N_R(x) = M_R\left(\frac{x}{\|S'\|_\infty}\right)$ and use monotonicity of $M_R$.

**Proof:** First, we prove (C) as (A) and (B) follow from (C). We have

$$S(u_m(x)) = \int_{\mathbb{R}} S'(\xi)\, \chi(\xi, u_m(x))\, d\xi$$

Test with any $\psi(x) \in L^\infty$ to get

$$\int_{B_R} \psi(x)\, S(u_m(x))\, dx = \int_{B_R}\int_{\mathbb{R}} S'(\xi)\, \chi(\xi, u_m(x))\, \psi(x)\, d\xi\, dx \qquad (*)$$

Unfortunately, $S'(\xi)\psi(x)\, \mathbb{1}_{B_R}(x)$ is not an $L^1$ function so we cannot use weak-$*$ convergence of $\chi(\xi, u_m(x))$.

We split the integral in $\xi$ over $[-k,k]$ and its complement.

$$\int_{B_R}\int_{\mathbb{R}} S'(\xi)\, \chi(\xi, u_m(x))\, \psi(x)\, d\xi\, dx = \underbrace{\int_{B_R}\int_{-k}^{k}}_{=:A_{m,k}} + \underbrace{\int_{B_R}\int_{\mathbb{R}\setminus[-k,k]}}_{=:B_{m,k}}$$

Using tail estimate we need to control the second term uniformly in $k$.

We estimate $\left| \overbrace{\int_{B_R} \int_{\mathbb{R}\setminus[-k,k]} S'(\xi)\, \Psi(x)\, \chi(\xi, u_n(x))\, d\xi\, dx}^{B_{n,k}} \right|$

$$\leq \int_{B_R} \int_{\mathbb{R}\setminus[-k,k]} \|S'\|_\infty \|\Psi\|_\infty |\chi(\xi, u_n(x))|\, d\xi\, dx$$

OBSERVATION: derivative of $\xi \mapsto (|\xi| - k)^+$

$$= \|S'\|_\infty \|\Psi\|_\infty \int_{B_R} \int_{\mathbb{R}\setminus[-k,k]} \operatorname{sgn} \xi \cdot \chi(\xi, u_n(x))\, d\xi\, dx =$$

$$= \|S'\|_\infty \|\Psi\|_\infty \int_{B_R} (|u_n(x)| - k)^+\, dx =$$

$$= \|S'\|_\infty \|\Psi\|_\infty \int_{B_R \cap |u_n| \geq k} |u_n(x)|\, dx \leq \|S'\|_\infty \|\Psi\|_\infty\, \omega_R(k)$$

As $\{u_n\}$ i weakly compact in $L^1$, $\omega_R(k) \to 0$ as $k \to \infty$.
It follows that

$$\lim_{k\to\infty} \lim_{n\to\infty} B_{n,k} = 0.$$

For term $A_{n,k}$ we can apply weak convergence so that

$$\lim_{k\to\infty} \lim_{n\to\infty} A_{n,k} = \int_{B_R} \int_{\mathbb{R}} S'(\xi)\, f(x,\xi)\, \Psi(x)\, dx\, d\xi$$

(for the second limit we used dominated convergence as we know that $f(x,\xi)$ is integrable on $B_R \times \mathbb{R}$).

Coming back to (*)

$$\int_{B_R} \psi(x) \operatorname*{w-lim}_{n\to\infty} S(u_n(x))\,dx = \int_{B_R}\int_{\mathbb{R}} S'(\xi)\psi(x) f(x,\xi)\,d\xi\,dx$$

for all $\psi \in L^\infty(B_R)$ (in particular $C_c^\infty$) and all $R \geq 0$. It follows that

$$\operatorname*{w-lim}_{n\to\infty} S(u_n) = \int_{\mathbb{R}} S'(\xi) f(x,\xi)\,d\xi$$

as desired :). This proves (A), (B) and (C).

Now, we prove (D). Fix $x$ and note that (A) implies that $u(x) = \int_{\mathbb{R}} f(x,\xi)\,d\xi$. Hence, for any $S$ with $S' \in L^\infty$ and $S$ convex

$$\operatorname*{w-lim}_{n\to\infty} S(u_n(x)) = \int S'(\xi) f(x,\xi)\,d\xi \geq$$

$$\geq \int S'(\xi)\,\chi(\xi, u(x))\,d\xi = S(u(x)).$$

by variational formulation.

Consider three sentences:

(X) $u_n \to u$ strongly

(Y) for all $S$ convex with $S' \in L^\infty$, $S(0)=0$, $S(u_n) \to S(u)$.

(Z) $f(x,\xi) = \chi(\xi, u(x))$.

Clearly (X) $\Leftrightarrow$ (Y) follows from standard theory.

(Y) $\Rightarrow$ (Z). Suppose that $f(x,\xi) \neq \chi(\xi, u(x))$. Then, inequality above has to be strict for $S(\xi) = \xi^2$ which is strongly convex. Contradiction with (Y).

(Z) $\Rightarrow$ (Y). Follows directly from representation of the weak limit in terms of kinetic function. $\blacksquare$

**THEOREM 4** (**Existence of Young measure again**).
Let $\{\nu_x\}_x$ be a family of probability measures as above. Then $\{\nu_x\}_x$ coincides with Young measures $\{\mu_x\}_x$ generated by $\{u^\varepsilon\}_{\varepsilon>0}$.

**Proof:** For any $S(\xi)$, the function $S(\xi) - S(0)$ vanishes at $\xi = 0$. It follows that

$$\underset{\varepsilon \to 0}{w\text{-}\lim} \left[ S(u^\varepsilon) - S(0) \right] = \underset{\varepsilon \to 0}{w\text{-}\lim}\, S(u^\varepsilon) - S(0)$$

$$\underset{\varepsilon \to 0}{w\text{-}\lim} \left[ S(u^\varepsilon) - S(0) \right] = \int S'(\xi) f(x,\xi)\, d\xi = -\left[ S(0) - \int S(\xi)\, d\nu_x(\xi) \right]$$

$$\underset{\varepsilon \to 0}{w\text{-}\lim}\, S(u^\varepsilon) - S(0) = \int S(\xi)\, d\mu_x(\xi) - S(0)$$

here, we used that $\frac{\partial}{\partial \xi} f(x,\xi) = \delta_0 - \nu_x$

It follows that $\int S(\xi)\, d\nu_x(\xi) = \int S(\xi)\, d\mu_x(\xi)$. $\blacksquare$

**Remark:** This is equivalent construction of YMs by kinetic functions.