# Hyperbolic Conservation Laws

# Tutorial

## Topic 8: Kinetic approximations for SCL.

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# Topic 8: Kinetic approximation for SCL

In the lecture it was proved that there is a solution to the kinetic approximation

$$\frac{\partial}{\partial t} f_\lambda(t,x,\xi) + a(\xi)\cdot\nabla_x f_\lambda(t,x,\xi)$$

$$+ \lambda\left[f_\lambda(t,x,\xi) - \chi(\xi; u_\lambda(t,x))\right] = 0$$

where $u_\lambda(t,x) = \int_{\mathbb{R}} f_\lambda(t,x,\xi)\,d\xi$ and initial condition for $f_\lambda(0,x,\xi) =: f^0(x,\xi)$ is given.

The solution was constructed in the Banach space

$$Y_T = C([0,T]; L^1_{x,\xi})$$

We also let $X_T = C([0,T]; L^1_x)$.

**Problem 1:** Existence was proved with BFPT. For $v\in X_T$ we solve

$$\frac{\partial}{\partial t} f_\lambda + a(\xi)\cdot\nabla_x f_\lambda + \lambda\left(f_\lambda - \chi(\xi, v(t,x))\right) = 0$$

Then, we define $\Phi(v) = \int_{\mathbb{R}} f_\lambda(t,x,\xi)\,d\xi$.

(we keep initial condition $f_\lambda|_{t=0} = f_0(x,\xi)$).

This is well-defined: we know that for linear problem

$$\frac{\partial}{\partial t} f + \underbrace{a(\xi)}_{\in L^\infty_{loc}}\cdot\nabla_x f + \lambda f = \underbrace{g}_{L^1_{t,x,\xi}} \qquad (*)$$

$$f|_{t=0} = f_0 \in L^1$$

there is a solution $f \in Y_T$, in particular $\int f\,d\xi$ belongs to $X_T$ and the map is well-defined.

**LEMMA.** $\Phi : X_T \to X_T$ is contractive.

**PROOF.** Let $v_1, v_2 \in X_T$ and $f_1, f_2 = \Phi(v_1), \Phi(v_2)$. Then $f_1 - f_2$ solves

$$\partial_t(f_1 - f_2) + a(\xi)\cdot\nabla_x(f_1 - f_2) + \lambda(f_1 - f_2) =$$

$$= \lambda\left(\chi(\xi, v_1) - \chi(\xi, v_2)\right).$$

We know from the theory of linear equation that

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} |f| + \lambda \int_{\mathbb{R}^d\times\mathbb{R}} |f| \leq \int_{\mathbb{R}^d\times\mathbb{R}} |g| .$$

Using this with $g = \chi(\xi, v_1) - \chi(\xi, v_2)$ we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} |f_1-f_2| + \lambda\int_{\mathbb{R}^d\times\mathbb{R}} |f_1-f_2| \leq \lambda\int_{\mathbb{R}^d\times\mathbb{R}} |\chi(\xi,v_1)-\chi(\xi,v_2)|$$

$$\leq \lambda\int_{\mathbb{R}^d} |v_1(t,x)-v_2(t,x)| \quad \text{(by contraction property)}$$

$$\leq \lambda \|v_1-v_2\|_{X_T} .$$

Multiply this with $e^{\lambda t}$ to get

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} e^{\lambda t}|f_1-f_2| \leq \lambda e^{\lambda t} \|v_1-v_2\|_{X_T} .$$

$$\Rightarrow e^{\lambda T}\int_{\mathbb{R}^d\times\mathbb{R}} |f_1-f_2|\, dx\, d\xi \leq \|v_1-v_2\|_{X_T} \int_0^T \lambda e^{\lambda t}\, dt$$

$$= \|v_1-v_2\|_{X_T} (e^{\lambda T}-1)$$

$$\Rightarrow \int_{\mathbb{R}^d\times\mathbb{R}} |f_1-f_2| \leq (1-e^{-\lambda T})\, \|v_1-v_2\|_{X_T}$$

But $$\int_{\mathbb{R}^d} |u_1-u_2| = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} f_1\, d\xi - \int_{\mathbb{R}} f_2\, d\xi \right| dx \leq$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}} |f_1-f_2|\, d\xi\, dx$$

$$\Rightarrow \|u_1-u_2\|_{X_T} \leq (1-e^{-\lambda T})\, \|v_1-v_2\|_{X_T}. \qquad \blacksquare$$

**Problem 2: $L^\infty$ estimate in the proof.** Let

$$A_\infty := \inf\{A>0 : f(x,\xi)=0 \quad \forall\, |\xi|\geq A\}$$

**LEMMA.** If $f_0 \in L^1_{x,\xi}$ and satisfies sign property then

- $\|u(t,x)\|_\infty \leq A_\infty$
- $f(t,x,\xi)=0 \quad \forall\, |\xi|>A_\infty$.

**PROOF:** We first prove that the subset of $X_T$ given with $\|v(t,x)\|_\infty \leq A_\infty$ is invariant under $\Phi$.

Indeed, if $\|v\|_\infty \leq A_\infty$, then representation formula for $f$ is

$$f(t,x,\xi) = \overbrace{f^0(x-a(\xi)t,\xi)}^{=0 \text{ for } |\xi|>A_\infty} e^{-\lambda t} +$$

$$+ \lambda \int_0^t e^{-\lambda s} \underbrace{\chi(\xi; \underbrace{v(t-s, x-a(\xi)s)}_{\leq A_\infty})}_{=0 \text{ for } |\xi|>A_\infty} ds.$$

so we get $f(t,x,\xi)=0$ for $|\xi|>A_\infty$. Using the sign property (representation formula shows that if ID satisfies the sign property, the same is true for the solution),

$$u(t,x) = \overbrace{\int_0^\infty f(t,x,\xi)\,d\xi}^{\geq 0} - \overbrace{\int_{-\infty}^0 |f(t,x,\xi)|\,d\xi}^{\geq 0}$$

$$\Rightarrow |u(t,x)| \leq \max\left(\int_0^\infty f(t,x,\xi)\,d\xi, \int_{-\infty}^0 f(t,x,\xi)\,d\xi\right).$$

As $|f| \leq 1$ and $f$ is supported on $|\xi| \leq A_\infty$ we obtain that $\|v\|_\infty \leq A_\infty$ is invariant for $\Phi$.

Now, there is general simple fact that if one uses BFPT for $\Phi : X_T \to X_T$ and $X_T$ has closed subspace $Z$ invariant under $\Phi$ then the fixed point belongs to $Z$ (otherwise we can apply BFPT for $\Phi : Z \to Z$ here we use invariance) and get contradiction with uniqueness of the fixed point. $\blacksquare$

## Problem 3: Corollary 3.6.2

**LEMMA:** Kinetic approximation equation can be written as

$$\frac{\partial}{\partial t} f_\lambda + a(\xi) \cdot \nabla_x f_\lambda = \frac{\partial}{\partial \xi} m_\lambda(t,x,\xi)$$

for $m_\lambda$ nonnegative and bounded. Moreover, it satisfies the following estimates:

(i) for all convex $S$ with $S'$ bounded and $S(0) = 0$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} S''(\xi)\, m_\lambda(t,x,\xi) \le \int_{\mathbb{R}^d} \int_{\mathbb{R}} S'(\xi) f_0(x,\xi)\, d\xi\, dx,$$

(ii) for all $\xi \in \mathbb{R}$

$$\int_0^\infty \int_{\mathbb{R}^d} m_\chi(t,x,\xi)\, dx\, dt \leq \mu^0(\xi) \in C_0(\mathbb{R})$$

bold fcns vanishing at $\pm\infty$.

where

$$\mu^0(\xi) = \mathbb{1}_{\xi>0} \|f_0\|_{L^1(\mathbb{R}^d \times (\xi,\infty))} + \mathbb{1}_{\xi\leq 0} \|f_0\|_{L^1(\mathbb{R}^d \times (-\infty,\xi))}$$

and $m_\chi(t,x,\xi) = 0$ for $|\xi| > A_\infty$.

**PROOF:** Recall that in the first tutorial on kinetic formulation we proved there is $m(\xi)$ s.t.

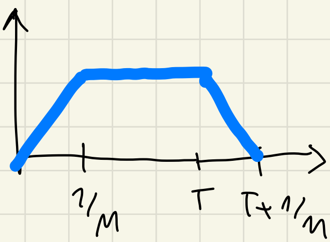
$$\chi(\xi,u) - f_\chi(t,x) = \frac{\partial}{\partial \xi} m(\xi), \quad m \in C_0(\mathbb{R}), \; m \geq 0$$

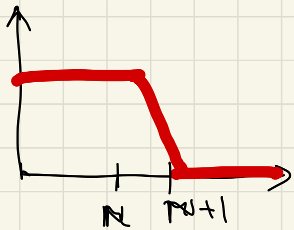
with $u = \int f(t,x)\, dt\, dx$. Hence, existence of $m_\chi$ is clear. To obtain the first bound we multiply eq$^n$ with $S'(\xi)$ to get

$$S'(\xi)\frac{\partial}{\partial t} f_\lambda + S'(\xi)\, a(\xi)\cdot\nabla_x f_\lambda = \frac{\partial}{\partial \xi} m_\lambda \cdot S'(\xi)$$

Formally: $\displaystyle\int \frac{\partial}{\partial \xi} m_\lambda \cdot S'(\xi) = -\int m_\lambda(t,x,\xi)\, S''(\xi)$

We consider test functions

$\varphi_M(t) =$ 


$\varphi_N(|x|) =$ 


Then:

(1) $\displaystyle -\int \partial_t \varphi_M(t)\, S'(\xi)\, f_\lambda(t,x,\xi)\, dt$

$\displaystyle = -\fint_0^{1/M} S'(\xi)\, f_\lambda(t,x,\xi)\, dt \longrightarrow -S'(\xi)\, f_0(x,\xi),$

(2) $\displaystyle\int_{|x|\geq N} S'(\xi)\, a(\xi)\, f_\lambda \cdot \nabla_x \varphi_N \longrightarrow 0$ by integrability

$$\Rightarrow \int_0^T\!\!\iiint m_\lambda(t,x,\xi)\, S''(\xi)\, dt\, dx\, d\xi \leq \int S'(\xi)\, f_0(x,\xi)\, dx\, d\xi - \int S'(\xi)\, f(T,x,\xi)\, dx\, d\xi$$

It remains to see that $-\int S'(\xi) f(T,x,\xi) d\xi \leq 0$.

From $\chi(\xi, u(t,x)) - f(t,x,\xi) = \frac{\partial}{\partial \xi} m(t,x,\xi)$ we get

$$S(u(t,x)) - \int S'(\xi) f(t,x,\xi) d\xi = -\int S''(\xi) m(t,x,\xi)$$

so the estimate is satisfied for nonnegative $S$.

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} S''(\xi)\, m_\lambda(t,x,\xi) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}} S'(\xi) f_0(x,\xi)\, d\xi\, dx$$

(by sending $T \to \infty$ after using $\int S'(\xi) f(T,x,\xi) \geq 0$).

Using point (ii) we know that $m_\lambda$ vanishes for $|\xi| > A_\infty$. Hence $|S(\xi)| \leq \|S'\|_\infty \cdot |\xi|$ so we can consider $\tilde{S}(\xi) = S(\xi) + \|S'\|_\infty A_\infty$ and get the result for all $S$ without nonnegativity assumption

**REMARK:** Note that it is not possible to send $T \to \infty$ in the definition of $\mathcal{E}_M(t)$ as $f_\lambda \notin L^1_{t,x,\xi}$ (it is "only" $C_t L^1_{x,\xi}$).

(proof ctd) As in the lecture, fix $\xi_0 > 0$ and consider $S(\xi) = (\xi - \xi_0)^+$ so that $S'(\xi) = \mathbb{1}_{\xi > \xi_0}$ and $S''(\xi) = \delta_{\xi = \xi_0}$. Note that $S$ is nonnegative

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}^d} m_\lambda(t, x, \xi_0) \, dt \, dx \leq$$

$$\leq \int_{\mathbb{R}^d} \int_{\xi > \xi_0} f_0(x, \xi) \, dx \, d\xi = \| f_0 \|_{L^1(\mathbb{R}^d \times (\xi_0, \infty))}$$

For $\xi_0 < 0$ we choose $S(\xi) = (\xi - \xi_0)^-$ so that $S'(\xi) = -\mathbb{1}_{\xi < \xi_0}$ and $S''(\xi) = \delta_{\xi = \xi_0}$. Note again that $S$ is nonnegative. Hence,

$$\int_0^\infty \int_{\mathbb{R}^d} m_\lambda(t, x, \xi_0) \, dt \, dx \leq$$

$$\leq \int_{\mathbb{R}^d} \int_{\xi < \xi_0} - f_0(x, \xi) \, dx \, d\xi = \int_{\mathbb{R}^d} \int_{\xi < \xi_0} |f_0| \, dx \, d\xi .$$

Finally, for $|\xi_0| > A_\infty$ we take $S$ supported for $|\xi| > A_\infty$ and strictly convex and $S \geq 0$. Using bound above we get

$$\underbrace{\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \underbrace{S''(\xi)}_{>0 \text{ for } |\xi| > A_\infty} \overbrace{m_\lambda(t,x,\xi)}^{\geq 0} \, d\xi \, dx \, dt}_{\geq 0} \leq \underbrace{\iint f_0(x,\xi) S'(\xi) \, d\xi \, dx}_{=0}$$

$$\Rightarrow \iiint_{|\xi| > A_\infty} S''(\xi) \, m_\lambda(t,x,\xi) = 0. \text{ But } S''(\xi) > 0 \text{ so}$$

that $m_\lambda = 0$ for $|\xi| > A_\infty$. ∎