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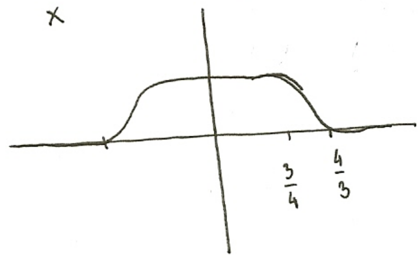
$H^s$ - Sobolev space on $\mathbb{R}^n$

$$\|uv\|_{H_s} \lesssim \|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s} \quad , \quad s \geq 0$$

Littlewood - Paley decomposition: (I)



$$\varphi(\xi) = \chi\left(\frac{\xi}{2}\right) - \chi(\xi)$$

$$\operatorname{supp}\varphi \subseteq \left\{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$$

$$(*)\ \chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1$$

Dyadic blocks: $\widehat{\Delta_j u}(\xi) = \hat{u}(\xi)\varphi(2^{-j}\xi) \quad , \; j \geq 0$

$$\widehat{S_0 u}(\xi) = \widehat{\Delta_{-1} u}(\xi) = \hat{u}(\xi)\chi(\xi)$$

$$\widehat{S_j u}(\xi) = \chi(2^{-j}\xi)\hat{u}(\xi)$$

$$(*) \quad \sum_{j\leq 1}\Delta_j u = u \quad \text{on} \quad S'(\mathbb{R}^n)$$

$$\operatorname{supp}\Delta_j u \subset 2^j C\left(0, \frac{3}{4}, \frac{8}{3}\right)$$

$$\Rightarrow \quad \Delta_j\Delta_{j'}u \equiv 0 \qquad \text{if } |j-j'| > 2$$

$$\mathcal{F}(S_{j-1}u\ \Delta_j v) \subset 2^j\, C(0,r,R)$$

(II) Functional space:

$$\|u\|_{H^s}^2 = \int(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi \approx \sum_j 2^{2js}\|\Delta_j u\|_{L^2}^2$$

Generalization: Besov spaces

$$\|u\|_{B^s_{p,q}} = \left\| 2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^n)}\right\|_{l^q(j\geq 1)}$$

Example:

$$B^s_{2,2} = H^s \qquad s \in (0,1)$$

$$B^s_{\infty,\infty} = C^s$$

Lemma: Let $(u_j)_{j\geq -1} \in C^\infty$ s.t. $\operatorname{supp}\hat{u}_j \subset 2^j C(0,r,R)$ , $j \geq 0$

$$\operatorname{supp}\hat{u}_{-1} \subset B(0,R)$$

Assume that $C(u) := \left\| 2^{js}\|u_j\|_{L^p}\right\|_{l^q} < \infty$

Then $\sum u_j \in B^s_{p,q}$ and $\|\sum u_j\|_{B^s_{p,q}} \simeq C(u)$



Sketch of proof:

$$\|\Delta_j(\sum_{j'} u_{j'})\|_{L^p} = \sum_{|j'-j|\le N} \|\Delta_j u_{j'}\|_{L^p}$$

$$\Delta_j u_{j'} = 2^{jn} \mathcal{F}^{-1}\varphi(2^j\ ) * u_{j'}$$

$$\|\Delta_j u_{j'}\|_{L^p} \le C \|u_{j'}\|_{L^p}$$

(III) Bernstein inequalities:

Take $u$ - smooth s.t. $\operatorname{supp}\hat{u} \subset B(0,R\lambda)$ then

$$\|u\|_{L^q} \lesssim C_R \lambda^{\frac{n}{p}-\frac{n}{q}} \|u\|_{L^p}, \quad 1 \le p \le q \le \infty$$

Proof: scaling $\rightarrow$ reduce $\lambda = 1$

Take $\Theta \in C_0^\infty(\mathbb{R}^n)$, $\Theta \equiv 1$ on $B(0,R)$

$\hat{u} = \Theta \hat{u}$

$\Rightarrow$ $u = \mathcal{F}^{-1}\Theta * u$

$$\|u\|_{L^q} \le \|\mathcal{F}^{-1}\Theta\|_{L^r} \|u\|_{L^p}$$

$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ because $q \ge p$

Corollary: $B^s_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s-(\frac{n}{p}-\frac{n}{q})}_{q,r}(\mathbb{R}^n)$ if $q \ge p$

Proof: Apply Bernstein to $\Delta_j u$ ( $u \in B^s_{p,r}$ ), $\lambda = 2^j$

$$\|\Delta_j u\|_{L^q} \lesssim 2^{j(\frac{n}{p}-\frac{n}{q})} \|\Delta_j u\|_{L^p}$$

Reverse Bernstein inequality: $\operatorname{supp}\hat{u} \subset C(0, r\lambda, R\lambda)$

$$\|\nabla u\|_{L^p} \approx \lambda \|u\|_{L^p}$$

Proof: $\Theta \equiv 1$ on $C(0, r, R)$. Assume $\lambda = 1$

$$\hat{u} = \sum_{k=1}^n \underbrace{\frac{-i\xi_k}{|\xi|^2}\Theta(\xi)}_{\hat{g}_k} \widehat{\partial_k u}(\xi)$$

$$\Rightarrow \|u\|_{L^p} \le \sum_{k=1}^n \|g_k\|_{L^1} \|\partial_k u\|_{L^p}$$

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(IV) Paraproduct

Formal idea: $u, v \in \mathcal{S}'$

$$uv = \sum_{j,k} \Delta_j u \, \Delta_k v$$

$$= \underbrace{\sum_{j \le k-2} \Delta_j u \, \Delta_k v}_{I} + \sum_{k \le j-2} \cdots + \sum_{|j-k| \le 1}$$

$$\sum_{j \le k-2} \Delta_j u = S_{k-1} u \qquad \left( \widehat{S_{k-1} u} = \chi(2^{-(k-1)} \cdot) \hat{u} \right)$$

$I$ is called paraproduct of $u$ by $v$, denoted by $T_u v$

$$T_u v = \sum_j S_{j-1} u \, \Delta_j v$$

Remainder term: $R(u,v) = \sum_j \Delta_j u \, (\Delta_{j-1} + \Delta_j + \Delta_{j+1}) v$

~~Finally:~~ Formally: $uv = \underbrace{T_u v + T_v u}_{\substack{\text{always defined} \\ \text{in } S'}} + R(u,v)$

Important remark: $\operatorname{Supp} \mathcal{F}(S_{j-1} u \, \Delta_j v) \subset 2^j C(0, r, R)$

$$\operatorname{supp} \mathcal{F}(\Delta_j u \, \Delta_j v) \subset 2^j B(0,R)$$

Continuity results for paraproduct:

$$\| T_u v \|_{B^{s-t}_{p,q}} \lesssim \| u \|_{B^{-t}_{\infty,\infty}} \| v \|_{B^s_{p,q}}$$

$\forall s \ \forall p \ \forall q \ \forall t > 0$

$$\| T_u v \|_{B^s_{p,q}} \lesssim \| u \|_{L^\infty} \| v \|_{B^s_{p,q}}$$

Proof: According to lemma + important remark, it suffices to prove suitable estimates for $\| S_{j-1} u \, \Delta_j v \|_{L^p}$

$$\| S_{j-1} u \, \Delta_j v \|_{L^p} \le \| S_{j-1} u \|_{L^\infty} \| \Delta_j v \|_{L^p}$$

$\rightarrow$ $u \in L^\infty$ $\quad S_{j-1} u = \underbrace{2^{(j-1)n} (\mathcal{F}^{-1} \chi)(2^{(j-1)} \cdot)}_{\substack{L^1 \\ \text{(independent of } j)}} * \underbrace{u}_{L^\infty}$

$\rightarrow$ $u \in B^{-t}_{\infty,\infty}$

$$\| S_{j-1} u \|_{L^\infty} = \sum_{k \le j-2} \left( \| \Delta_k u \|_{L^\infty} 2^{-kt} \right) 2^{kt} \le \| u \|_{B^{-t}_{\infty,\infty}} \left( \sum_{k \le j-2} 2^{kt} \right)$$

Continuity for remainder term:

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,q}} \lesssim \|u\|_{B^{s_1}_{p_1,q_1}} \|v\|_{B^{s_2}_{p_2,q_2}}$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \quad ; \quad 1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty, \quad s_1 + s_2 > 0$$

Proof:

$$R(u,v) \approx \sum_j \Delta_j u \, \Delta_j v \qquad \mathcal{F}(\Delta_j u \, \Delta_j v) \subset B(0, 2^j R)$$

$$\|\Delta_j R(u,v)\|_{L^p} \leq \sum_{j' \geq j-3} \|\Delta_j (\Delta_{j'} u \, \Delta_{j'} v)\|_{L^p}$$

$$\leq C \sum_{j' \geq j-3} \|\Delta_{j'} u\|_{L^{p_1}} 2^{j's_1} \|\Delta_{j'} v\|_{L^{p_2}} 2^{j's_2}$$

$$2^{j(s_1+s_2)} \|\Delta_j R\|_{L^p} \lesssim \sum_{j' \geq j-3} \underbrace{2^{(s_1+s_2)(j-j')}}_{l^1} \underbrace{\|\Delta_{j'} u\|_{L^{p_1}} 2^{j's_1}}_{l^{q_1}} \underbrace{\|\Delta_{j'} v\|_{L^{p_2}} 2^{j's_2}}_{l^{q_2}}$$

because $s_1 + s_2 > 0$

Corollary: $\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{L^\infty} \|v\|_{B^s_{p,q}} + \|v\|_{L^\infty} \|u\|_{B^s_{p,q}} \quad \forall s > 0, \; 1 \leq p, q \leq \infty$

Proof:

$$uv = T_{\underset{L^\infty}{u}} \underset{B^s_{p,q}}{v} + R(\underset{B^0_{\infty,\infty}}{u}, \underset{B^s_{p,q}}{v}) + T_{\underset{L^\infty}{v}} \underset{B^s_{p,q}}{u}$$

$$\|u\|_{B^0_{\infty,\infty}} = \sup_j \|\Delta_j u\|_{L^\infty}$$

$$\|\Delta_j u\|_{L^\infty} \leq C \|u\|_{L^\infty} \Rightarrow \|u\|_{B^0_{\infty,\infty}} \leq C \|u\|_{L^\infty}$$

$$\|F(u)\|_{B^s_{p,q}} \leq \; ?$$