# Constructive fragment of the $\mu$–calculus

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Brno, August 21st 2010

1. Basics
   - The very basics
   - Main definitions

2. The big picture
   - Theorems about continuous fragment [G. Fontaine]
   - Other relations

# The very basics – syntax

## Modal $\mu$ syntax

$$\varphi \longrightarrow \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond \varphi \mid \mu p.\varphi$$

$p$ is a propositional letter (*Prop*)

$\mu p.\varphi$ is allowed when every occurrence of $p$ in $\varphi$ is in range of even number of negations.

# The very basics – syntax

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## Normal form

We set $\bot = \neg \top$, $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$,
$\Box\varphi = \neg \Diamond \neg\varphi$, $\nu p.\varphi = \neg\mu p.\neg\varphi[p := \neg p]$.
Thus we obtain a normal form for $\mu$–formulae by pushing negations as deep as it is possible. We get:

$$\varphi \longrightarrow \bot \mid \top \mid p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \Box \varphi \mid \mu p.\varphi \mid \nu p.\varphi$$

Where $\mu p.\varphi$ and $\nu p.\varphi$ are allowed when there are no $\neg p$ in $\varphi$.

# The very basics – semantics

## Kripke models

$\mathcal{M} = (M, R, V)$ is a Kripke model when $(M, R)$ is a graph and $V : Prop \to \mathcal{P}(M)$ assigns values to propositional letters.

We define $[\![\varphi]\!]_{\mathcal{M}}$ – a subset of $M$ of points in which $\varphi$ is true in a usual way. Recall that:

$$[\![\mu p.\varphi]\!]_{\mathcal{M}} = \bigcap\{A \subseteq M : [\![\varphi]\!]_{\mathcal{M}[p:=A]} \subseteq A\}$$

where $\mathcal{M}[p := A] = (M, R, V')$, $V'(p) = A$ and $V'(q) = V(q)$ for $q \neq p$.

# The very basics – fixpoints

## Finding fixpoints

Fix $\varphi$, $p$ and $\mathcal{M}$. There exists $\alpha$ such that

$$[\![\mu p.\varphi]\!]_{\mathcal{M}} = \varphi_p^{\alpha}(\varnothing)$$

where $\varphi_p^0(A) = [\![\varphi]\!]_{\mathcal{M}[p:=A]}$, $\varphi_p^{\beta+1}(A) = \varphi_p(\varphi_p^{\beta}(A))$, and for limit ordinals $\lambda$: $\varphi_p^{\lambda}(A) = \bigcup_{\beta<\lambda} \varphi_p^{\beta}(A)$.

1. Basics
   - The very basics
   - Main definitions

2. The big picture
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- bounded in $p$ iff there exists $k \in \omega$ such that for all models $\mathcal{M}$: $\varphi_p^{k+1}(\varnothing) = \varphi_p^k(\varnothing)$.
- constructive in $p$ iff for all models $\mathcal{M}$: $\varphi_p^{\omega+1}(\varnothing) = \varphi_p^\omega(\varnothing)$.

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- bounded in $p$ iff there exists $k \in \omega$ such that for all models $\mathcal{M}$: $\varphi_p^{k+1}(\emptyset) = \varphi_p^k(\emptyset)$.
- constructive in $p$ iff for all models $\mathcal{M}$: $\varphi_p^{\omega+1}(\emptyset) = \varphi_p^{\omega}(\emptyset)$.
- continuous in $p$ iff for all models $\mathcal{M} = (M, R, V)$ and all $s \in M$ the following are equivalent:
  - $s \in [\![\varphi]\!]_{\mathcal{M}}$,
  - there exists finite $F \subseteq V(p)$ such that $s \in [\![\varphi]\!]_{\mathcal{M}[p:=F]}$.

1 Basics
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- Main definitions

2 The big picture
- Theorems about continuous fragment [G. Fontaine]
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# Alternative definitions

## Preliminary notions:

- Fix $\mathcal{M} = (M, R, V)$. A family $\mathcal{F}$ of subsets of $M$ is directed when for all $A, B \in \mathcal{F}$, there exists $C \in \mathcal{F}$ such that $A \cup B \subseteq C$.

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- We call a family $\mathcal{O}$ of subsets of $M$ a Scott open in the powerset algebra $\mathcal{P}(M)$ iff it is closed under upset and such that for every directed family $\mathcal{F}$ satisfying $\bigcup \mathcal{F} \in \mathcal{O}, \mathcal{F} \cap \mathcal{O} \neq \varnothing$.

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- A formula is Scott continuous in $p$ iff for all models $\mathcal{M} = (M, R, V)$, the map $\varphi_p : \mathcal{P}(M) \to \mathcal{P}(M)$ is Scott continuous.

# More facts about continuous formulae

**Proposition**

A map $f$ is Scott continuous iff it preserves directed joins.

**Theorem**

A formula $\varphi$ is Scott continuous in $p$ iff it is continuous in $p$.

# Syntactic characterization of continuous fragment

## Syntactic characterization of continuous fragment [G. Fontaine]

Let $P \subseteq Prop$. The following grammar defines the set of formulas $CF(P)$:

$$\varphi \longrightarrow \top \mid p \mid \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \mu q.\chi$$

where $p \in P$, propositional letters from $P$ does not occur in $\psi$ and $\chi \in CF(P \cup \{q\})$.

## Theorem [G. Fontaine]

Let $P \subseteq Prop$. If $\varphi \in CF(P)$, then $\varphi$ is continuous in $p$ for every $p \in P$. If $\varphi$ is continuous in $p$, then $\varphi$ is equivalent to $\psi \in CF(p)$.

# Continuous are constructive

## Theorem

Every continuous formula in $p$ is constructive in $p$.

## Proof:

Suppose that $\varphi$ is continuous in $p$. Consider $\mathcal{F} = \{\varphi_p^i(\varnothing) : i \in \omega\}$. Note that $\mathcal{F}$ is a directed family and thus $\varphi$ must preserve $\bigcup_{i\in\omega} \varphi_p^i(\varnothing)$. Therefore $\varphi(\bigcup \mathcal{F}) = \bigcup \varphi[\mathcal{F}] = \bigcup_{i\in\omega} \varphi_p^i(\varnothing) = \bigcup_{i>0} \varphi_p^i(\varnothing) = \bigcup_{i\in\omega} \mathcal{F}$. Therefore $\bigcup \mathcal{F}$ is the least fixpoint of $\varphi_p$, thus $\varphi$ is constructive in $p$.

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# Continuous and bounded

## Continuous $\notin$ bounded

Consider $\varphi = \Diamond p \vee \Box \bot$. It is continuous in $p$ according to the syntactical characterization. But it is not bounded in $p$ since there may be arbitrary long paths ending on a leaf.

## Bounded $\notin$ continuous

Consider $\varphi = \Box p \wedge \Box\Box \bot$. It is bounded since $\varphi_p^2(\varnothing) = \varphi_p^1(\varnothing)$. But $\varphi$ is not continuous in $p$. To show it consider a model with a root 0 and children $1, 2, 3, \ldots$. Let $\mathcal{F} = \{\{1, \ldots, n\} : n > 0\}$ – it is a directed family and $\varphi(\bigcup \mathcal{F}) = \mathbb{N}$, but $\bigcup \varphi[\mathcal{F}] = \mathbb{N} - \{0\}$.

# The picture

**Constructive**

$\not\subseteq$ $\not\subseteq$

**Bounded** **Continuous**

# Motivation

**Yde Venema's conjecture**

Let $\varphi$ be a formula constructive in $p$. Then there exists a continuous formula $\psi$ such that $\mu p.\varphi \equiv \mu p.\psi$.

# Constructive formulae behave badly

## Proposition

Constructive formulae are not closed on disjunction.

## Proof...:

Consider: $\varphi = (\Diamond p \wedge q \wedge \Box q) \vee (\Box p \wedge \neg q \wedge \Box q)$ and $\psi = \Box \bot$. Let us observe that both formulae are bounded and hence constructive in $p$. For $\psi$ this is obvious, for $\varphi$ in all models there is: $\varphi_p^0(\varnothing) = [\![(\Box \bot \wedge \neg q)]\!]$ and
$\varphi_p^1(\varnothing) = [\![(\Diamond(\Box \bot \wedge \neg q \Box q) \wedge q \wedge \Box q) \vee (\Box(\Box \bot \wedge \neg q) \wedge \neg q \wedge \Box q)]\!] =$
$= [\![(\Box \bot \wedge \neg q)]\!]$.

# Constructive formulae behave badly (2)

**...Proof:**

$(\Diamond p \wedge q \wedge \Box q) \vee (\Box p \wedge \neg q \wedge \Box q) \vee \Box \bot$ fix after exactly $\omega + 1$ steps – hence is not constructive. In particular there is a model in which $\varphi \vee \psi$ fix after $\omega + 1$ steps.
Example...

# Where to seek for constructive formulae?

## Syntactic characterization of continuous fragment (again)

Let $P \subseteq Prop$. The following grammar defines the set of formulas $CF(P)$: $\varphi \longrightarrow \top \mid p \mid \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \mu q.\chi$, where $p \in P$, propositional letters from $P$ does not occur in $\psi$ and $\chi \in CF(P \cup \{q\})$.

## Constructions beyond continuous

- $\Box\varphi$,
- $\nu p.\varphi$,
- $\mu p.\varphi$, for discontinuous $\varphi$ in $p$.

# Are all constructive formulae bounded or continuous?

## Counterexample:

$\varphi = \Diamond \Diamond p \vee (\Box p \wedge \Box\Box \bot)$.

## Proof:

$\varphi$ is not continuous in $p$. Recall a model we used to show that $\Box p \wedge \Box\Box \bot$ is discontinuous.

$\varphi_p^0(\varnothing) = [\![\Box \bot]\!]$,

$\varphi_p^1(\varnothing) = [\![\Diamond \Diamond \Box \bot \vee \Box \Box \bot]\!]$,

$\varphi_p^2(\varnothing) = [\![\Diamond^4 \Box \bot \vee \Diamond^2 \Box^2 \bot \vee \Box^2 \bot]\!]$,

$\varphi_p^\omega(\varnothing) = [\![\bigvee_{i \in \omega} \Diamond^{2i} \Box^2 \bot]\!]$.

$\varphi_p^{\omega+1}(\varnothing) = \varphi_p^\omega(\varnothing)$.

# The bigger picture

**Constructive**

$\supsetneq$

**Bounded** $\cup$ **Continuous**

$\supsetneq$ $\supsetneq$

**Bounded** **Continuous**

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