How fast can the fixpoints in modal μ calculus be reached?

Marek Czarnecki

Department of Philosophy and Sociology
University of Warsaw

Department of Mathematics, Computer Science and Mechanics
University of Warsaw

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The very basics – syntax

Modal μ syntax

$$\varphi \longrightarrow \top \mid p \mid x \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond \varphi \mid \mu x.\varphi$$

p is a propositional letter (*Prop*), x is an individual variable (*Var*) and construction $\mu x.\varphi$ is allowed when every occurrence of x in φ is in range of even number of negations.

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Normal form

Set $\bot = \neg \top$, $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$,
$\Box\varphi = \neg \Diamond \neg\varphi$ and $\nu x.\varphi = \neg\mu x.\neg\varphi[x := \neg x]$.
Thus we obtain a normal form for μ formulae by pushing negations as deep as it is possible. We get:

$$\varphi \longrightarrow \bot \,|\, \top \,|\, p \,|\, \neg p \,|\, x \,|\, \neg x \,|\, \varphi \vee \varphi \,|\, \varphi \vee \varphi \,|\, \Diamond \varphi \,|\, \Box \varphi \,|\, \mu x.\varphi \,|\, \nu x.\varphi$$

Where $\mu x.\varphi$ and $\nu x.\varphi$ are allowed when there are no $\neg x$ in φ.

The very basics – semantics

Kripke models

$\mathcal{M} = (M, R, V)$ is a Kripke model when (M, R) is a graph and $V : Prop \to \mathcal{P}(M)$ assigns values to propositional letters.

A valuation (assignment) $\tau : Var \to \mathcal{P}(M)$ assigns values to individual variables.

We define $[\![\varphi]\!]_{\mathcal{M},\tau}$ – a subset of M of points in which φ is true in a usual way. Recall that:

$$[\![\mu x.\varphi]\!]_{\mathcal{M},\tau} = \bigcap\{A \subseteq M : [\![\varphi]\!]_{\mathcal{M},\tau[x:=A]} \subseteq A\}$$

where $\tau[x := A](x) = A$ and $\tau[x := A](y) = \tau(y)$ for $y \neq x$.

The very basics – fixpoints

Finding fixpoints

Fix φ, x, $\mathcal{M}$ and τ. There is α such that

$$[\![\mu x.\varphi]\!]_{\mathcal{M},\tau} = \varphi_x^{\alpha}(\varnothing)$$

where $\varphi_x^0(A) = [\![\varphi]\!]_{\mathcal{M},\tau[x:=A]}$, $\varphi_x^{\beta+1}(A) = \varphi_x(\varphi_x^{\beta}(A))$, and for limit ordinals λ: $\varphi_x^{\lambda}(A) = \bigcup_{\beta<\lambda} \varphi_x^{\beta}(A)$.

1. Basics
   - The very basics
   - The main concept
2. Fast and slow formulae
   - Some formulae never fix
   - Some formulae always reach their fixpoints fast
3. Can we control the number of iterations for ω and above?
   - Formulae with fuses
   - How do fuses work?
4. What next?

Definition

We say that a modal μ–formula φ fixes after α steps in x when α is the least ordinal number such that for all $\mathcal{M}$ and τ,

$$[\![\mu x.\varphi]\!]_{\mathcal{M},\tau} = \varphi_x^\alpha(\varnothing)$$

We denote it by $\mathcal{O}_x(\varphi) = \alpha$.

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What is it all about?

We investigate after which ordinal numbers of steps modal μ formulae may fix.

Examples

Examples:

$$\mathcal{O}_x(\Diamond x \vee p) = \omega$$

$$\mathcal{O}_x(x) = 0$$

An example

Fact

$\mathcal{O}_x(\Box x)$ is undefined i.e. $\Box x$ goes forever.

An example (2)

Proof:

We show that there are models in which $\Box x$ needs more than any given number of steps to fix.

We use an assignment of trees to ordinal numbers. To 0 we assign just a single point – a root. For $\alpha + 1$ we construct the tree by taking a new point to be the root and attaching to it the root of the tree for α. For limit ordinals λ we construct the tree by taking a new point to be the root and attach to it all the roots of trees constructed before $\alpha < \lambda$. It is easy to see that in a tree assigned to λ formula $\Box x$ fixes after $\lambda + 1$ steps – thus $\mathcal{O}_x(\Box x)$ is undefined.

Examples for $\alpha < \omega$

Fact

Let $\varphi_n = \Box x \wedge \Box^{n+1} \bot$.
Then $\mathcal{O}_x(\varphi_n) = n$.

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For $k \leq l$,

$$\Box^k \bot \wedge \Box^l \bot \equiv \Box^k(\Box^{l-k} \bot \wedge \bot) \equiv \Box^k \bot$$

and

$$\Box^k \bot \vee \Box^l \bot \equiv \Box^l \bot$$

Examples for $\alpha < \omega$ (2)

Fact

Let $\varphi_n = \Box x \wedge \Box^{n+1} \bot$.
Then $\mathcal{O}_x(\Box x \wedge \Box^{n+1} \bot) = n$.

Proof:

Fix $n \in \omega$, a model and a valuation
$\varphi_n^{n+1}(\emptyset) = [\![\bigvee_{i=0}^{n+1} (\Box^{i+1} \bot \wedge \Box^{n+1} \bot)]\!] = [\![\Box^{n+1} \bot \wedge \bigvee_{i=0}^{n+1} \Box^{i+1} \bot]\!] = [\![\Box^{n+1} \bot \wedge \Box^{n+2} \bot]\!] = [\![\Box^{n+1} \bot]\!] = \varphi_n^n(\emptyset)$.
This gives the upper bound for number of iterations of φ_n. For the lower bound consider models assigned to finite ordinals.

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Inspiration

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Our investigation is motivated by a question asked by Damian Niwiński whether there exists a formula which fixes after $\omega + 1$ steps and, in a broader sense, whether it is possible to control the number of iterations of formulae above ω.

Mikołaj Bojańczyk's conjecture

The formula $(\Diamond x \wedge \Box p_1 \wedge p_1) \vee (\Box x \wedge \Box p_1 \wedge \neg p_1) \vee \Box\bot$ fixes in $\omega + 1$ steps.

The main result

Mikołaj Bojańczyk conjecture is true. We generalize this result showing formulae that fix in α steps for all $\alpha < \omega^2$.

The formulae

Sets of fuses

For $n > 0$ and $0 \leq i \leq n$ let $C_i^n = \neg p_1 \wedge \cdots \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$.

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- $$\psi_{\omega \cdot n} = \bigvee_{i=0}^{n-1} (\Diamond x \wedge C_i^n \wedge \Box C_i^n) \vee \bigvee_{i=0}^{n-2} (\Box x \wedge C_{i+1}^n \wedge \Box C_i^n)$$

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- $$\psi_{\omega \cdot n+m} = \psi_{\omega \cdot n} \vee \bigvee_{i=0}^{m-1} (\Box x \wedge \bigwedge_{j=0}^{i} \Box^j C_n^n \wedge \Box^{i+1} C_{n-1}^n)$$

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- $$\psi_{\omega \cdot n+m} = \psi_{\omega \cdot n} \vee \bigvee_{i=0}^{m-1} (\Box x \wedge \bigwedge_{j=0}^{i} \Box^j C_n^n \wedge \Box^{i+1} C_{n-1}^n)$$
- $$\varphi_{\omega \cdot n+m} = \psi_{\omega \cdot n+m} \vee \Box \bot$$

The main lemma

Lemma

Let $k > 0$, $\omega \cdot k \le \alpha < \omega^2$. Then

$$(a \vDash p_k \wedge a \in [\![\mu x.\varphi_\alpha]\!]) \Rightarrow a \in \varphi_\alpha^{\omega \cdot k}(\varnothing)$$

The main lemma (2)

Proof...:

Fix $\mathcal{M} = (M, R, V)$, τ and $a \in M$. There exists β such that $a \in \varphi_\alpha^\beta(\emptyset)$. We proceed by induction on β.

The base step and limit steps are trivial.
We need to show that
$\forall k > 0 \forall \omega \cdot k \leq \alpha < \omega^2 \left[(a \models p_k \wedge a \in \varphi_\alpha^{\gamma+1}(\emptyset)) \Rightarrow a \in \varphi_\alpha^{\omega \cdot k}(\emptyset) \right]$.
Fix $k > 0$ and let $\alpha = \omega \cdot n + m$, for $n \geq k$ and $m \in \omega$. Let us assume that $a \models p_k$ and $a \in \varphi_\alpha^\beta(\emptyset)$. Since $\beta = \gamma + 1$ we have $a \in \varphi_\alpha(\varphi_\alpha^\gamma(\emptyset))$. By the definition of φ_α, one of the following cases must hold:

The main lemma (3)

...Proof...:

- $a \vDash \Box\bot$ – then $a \in \varphi_\alpha^0(\varnothing) \subseteq \varphi_\alpha^{\omega\cdot k}(\varnothing)$,
- $a \vDash C_l^n \wedge \Box C_l^n$ for some $l < k$ – since $a \vDash p_k$, and there exists t such that aRt and $t \in \varphi_\alpha^\gamma(\varnothing)$. Therefore $t \vDash C_l^n$ which implies $t \vDash p_{l+1}$. By the induction hypothesis, since $t \in \varphi_\alpha^\gamma(\varnothing)$ and $t \vDash p_{l+1}$, we know that $t \in \varphi_\alpha^{\omega\cdot(l+1)}(\varnothing)$. Because $\omega \cdot (l+1)$ is a limit ordinal, there exists $s \in \omega$ such that $t \in \varphi_\alpha^{\omega\cdot l+s}(\varnothing)$. Therefore $a \in \varphi_\alpha^{\omega\cdot l+s+1}(\varnothing) \subseteq \varphi_\alpha^{\omega\cdot(l+1)}(\varnothing) \subseteq \varphi_\alpha^{\omega\cdot k}(\varnothing)$,

The main lemma (4)

...Proof:

- $a \vDash C^n_{l+1} \wedge \Box C^n_l$ for some $l < k-1$ – since $a \vDash p_k$, and for all t if aRt, then $t \in \varphi^{\gamma}_{\alpha}(\varnothing)$. Fix such t, then $t \vDash C^n_l$ and therefore $t \vDash p_{l+1}$, so by the induction hypothesis $t \in \varphi^{\omega\cdot(l+1)}_{\alpha}(\varnothing)$. Thus $a \in \varphi^{\omega\cdot(l+1)+1}_{\alpha}(\varnothing) \subseteq \varphi^{\omega\cdot k}_{\alpha}(\varnothing)$ since $l < k-1$,
- In other cases, namely: $a \vDash \bigvee_{i=0}^{m-1}(\Box x \wedge \bigwedge_{j=0}^{i} \Box^j C^n_n \wedge \Box^{i+1} C^n_{n-1})$, $a \vDash C^n_n$ which means $a \vDash \neg p_i$ for $i = 1, \ldots, n$, but this is a contradiction since $k \leq n$ and we assumed that $a \vDash p_k$.

Corollaries of the main lemma

Global view on fuses:

If a point in which p_i is true is in the least fixpoint of φ_α, it has to be added to it fast, that is after at most $\omega \cdot i$ steps. After that number of iterations the *fuse* p_i melts and no more points in which p_i is true may be added to the fixpoint.

Corollaries of the main lemma (2)

Local view on fuses – an example:

Let us consider the formula:

$$\varphi_{\omega\cdot 2+3} = (\Diamond x \wedge C_0^2 \wedge \Box C_0^2) \vee (\Diamond x \wedge C_1^2 \wedge \Box C_1^2) \vee (\Box x \wedge C_1^2 \wedge \Box C_0^2) \vee$$
$$\vee (\Box x \wedge C_2^2 \wedge \Box C_1^2) \vee (\Box x \wedge C_2^2 \wedge \Box C_2^2 \wedge \Box^2 C_1^2) \vee$$
$$\vee (\Box x \wedge C_2^2 \wedge \Box C_2^2 \wedge \Box^2 C_2^2 \wedge \Box^3 C_1^2)$$

Recall that $C_0^2 = p_1 \wedge p_2$, $C_1^2 = \neg p_1 \wedge p_2$ and $C_2^2 = \neg p_1 \wedge \neg p_2$.

Corollaries of the main lemma (3)

Local view on fuses – an example (2):

Example...

The main theorem

Theorem:

For every $\alpha < \omega^2$: φ_α fixes after α steps.

Proof...:

If there exists $i \leq n$ such that $a \vDash p_i$, then, by lemma we know that $a \in \varphi_\alpha^{\omega \cdot i}(\varnothing) \subseteq \varphi_\alpha^{\omega \cdot n+m}(\varnothing) = \varphi_\alpha^{\alpha}(\varnothing)$, since $a \in \varphi_\alpha^{\alpha+1}(\varnothing)$.
Let us now assume that for $i = 1, \ldots, n$, $a \vDash \neg p_i$ holds. Since $a \in \varphi_\alpha(\varphi_\alpha^\alpha(\varnothing))$, then by the definition of φ_α, $m > 0$ and one of the following cases must hold:

- $a \vDash \Box \bot$ – then trivially $a \in \varphi_\alpha^\alpha(\varnothing)$.

The main theorem (2)

...Proof...:

- $a \vDash \bigwedge_{j=0}^{i} \Box^j C_n^n \wedge \Box^{i+1} C_{n-1}^n$, for some $i = 0, \ldots, m-1$ and for every t such that aRt, $t \in \varphi_\alpha^\alpha(\varnothing)$.
  We proceed by induction on i to show that if $a \vDash \bigwedge_{j=0}^{i} \Box^j C_n^n \wedge \Box^{i+1} C_{n-1}^n$, then $a \in \varphi_\alpha^{\omega \cdot n+i+1}(\varnothing)$.
  For the base step let us assume that $i = 0$. Then for all t such that aRt, $t \vDash C_{n-1}^n$ holds. Therefore $t \vDash p_n$ and by the main lemma, $t \in \varphi_\alpha^{\omega \cdot n}(\varnothing)$. Thus $a \in \varphi_\alpha^{\omega \cdot n+1}(\varnothing)$.
  Suppose now that for $0 \leq i < k \leq m$ if $a \vDash \bigwedge_{j=0}^{i} \Box^j C_n^n \wedge \Box^{i+1} C_{n-1}^n$, then $a \in \varphi_\alpha^{\omega \cdot n+i+1}(\varnothing)$. We show that for $i = k$ this implication holds as well. Suppose that $a \vDash \bigwedge_{j=0}^{k} \Box^j C_n^n \wedge \Box^{k+1} C_{n-1}^n$ then for every t such that aRt, $t \vDash \bigwedge_{j=0}^{k-1} \Box^j C_n^n \wedge \Box^k C_{n-1}^n$ holds. Therefore, by the induction hypothesis $t \in \varphi_\alpha^{\omega \cdot n+k}(\varnothing)$, and thus $a \in \varphi_\alpha^{\omega \cdot n+k+1}(\varnothing)$.
  Hence, for every such case $a \in \varphi_\alpha^{\omega \cdot n+m}(\varnothing) = \varphi_\alpha^\alpha(\varnothing)$.

The main theorem (3)

...Proof...:

- In other cases, namely when $a \vDash \bigvee_{i=0}^{n-1}(\Diamond x \wedge C_i^n \wedge \Box C_i^n) \vee \bigvee_{i=0}^{n-2}(\Box x \wedge C_{i+1}^n \wedge \Box C_i^n)$ also $a \vDash C_i^{n+1}$ holds, for some $i = 1, \ldots n-1$. This means that $a \vDash p_n$ which is a contradiction, since we assumed that $a \vDash \neg p_n$.

  This shows that

$$\varphi_\alpha^{\alpha+1}(\varnothing) = \varphi_\alpha^\alpha(\varnothing)$$

The main theorem (4)

...Proof:

For $\alpha < \omega^2$ we can construct models in which φ_α fixes after α steps.

Open (?) questions

- Are there basic modal formulae that fix after ω^2 or more steps?

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- Are there basic modal formulae that fix after ω^2 or more steps?
- Are there μ–formulae that fix after ω^2 or more steps?
- Is there a formula that behaves as an ω–counter (at least in big enough models), that allows us to count uses of $\Box$ up to ω?
- Is it decidable whether $\mathcal{O}_x(\varphi)$ is defined, given φ?

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THANK YOU FOR YOUR ATTENTION