Semantics Coded by Coprimality in Finite Models

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Gent, 19 February 2009

Motivations

1. J. Mycielski — Analysis Without Actual Infinity,
   - Natural interpretation of basic notions of mathematical analysis (convergence; continuity, differentiability) in potentially infinite worlds.

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   - Representing syntax and semantics of arithmetics in potentially infinite worlds.

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   - Representing syntax and semantics of arithmetics in potentially infinite worlds.
3. M. Krynicki, K. Zdanowski — Theories of Arithmetics in Finite Models,
   - Investigating how certain arithmetics work in finite models – arithmetic of multiplication.

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   - Representing relations and interpreting addition and multiplication in language of division.

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   - Representing relations and interpreting addition and multiplication in language of division.
5. M. Mostowski, K Zdanowski — Coprimality in Finite Models,
   - Investigating how arithmetic of coprimality works in potentially infinite worlds.

Contents

FM–domain – a mathematical model of potentially infinite world

Definition

$\mathcal{A} = (\omega, R_1, ..., R_k)$ is a model purely relational model. FM–domain over $\mathcal{A}$ is a family of finite models $\mathrm{FM}(\mathcal{A}) = \{\mathcal{A}_n : n = 1, 2, ...\}$, where $\mathcal{A}_n = (\{0, ..., n-1\}, R_1^{(n)}, ..., R_k^{(n)})$.

Definition

Sentence ψ is true in $\mathrm{FM}(\mathcal{A})$, when it is true in almost all models from the FM–domain. $\mathrm{FM}(\mathcal{A}) \models_{sl} \psi \equiv \exists k \forall n > k \mathcal{A}_n \models \psi$.

Actual and potential infinity

Actual infinity	Potential infinity
$\{0, 1, 2, 3, 4 \ldots\}$	$\{0\}$
	$\{0, 1\}$
	$\{0, 1, 2\}$
	$\vdots$

Actual and potential infinity

Actual infinity	Potential infinity
$(\{0,1,2,3\dots\},s,\times,0)$	$(\{0\},s,\times,0)\models\neg\varphi$
$\models$	$(\{0,1\},s,\times,0)\models\neg\varphi$
$\exists^{\geq 2}x\exists y\,(y\times s(s(0))=x)$	$(\{0,1,2\},s,\times,0)\models\varphi$
$=\varphi$	$(\{0,1,2,3\},s,\times,0)\models\varphi$
	$\vdots$
$(\{0,1,2,3\dots\},s,\times,0)\models\varphi$	$\mathrm{FM}((\{0,1,2,3\dots\},s,\times,0))\models_{sl}\varphi$

FM–representability

A relation $R \subseteq \omega^r$ is FM–represented by a formula $\varphi(x_1, \dots, x_k)$ in FM–domain $\mathrm{FM}(\mathcal{A})$ iff, for any $a_1, \dots, a_k \in \omega$ the following conditions hold:

1. $\mathrm{FM}(\mathcal{A}) \models_{sl} \varphi[a_1, \dots, a_k] \iff R(a_1, \dots, a_k)$,
2. $\mathrm{FM}(\mathcal{A}) \models_{sl} \neg\varphi[a_1, \dots, a_k] \iff \neg R(a_1, \dots, a_k)$.

A relation R is FM–representable in $\mathrm{FM}(\mathcal{A})$ if there is a formula φ, which FM–represents R in $\mathrm{FM}(\mathcal{A})$.

FM–representability

FM–representability theorem (M. Mostowski)

A relation $R \subseteq \omega^r$ is FM–representable in $\mathrm{FM}(\mathcal{N})$ iff $R \in \Delta^0_2$.
$\mathcal{N} = (\omega, s, +, \times, <, 0, 1)$ – standard model of arithmetic.

Short characteristics of Δ^0_2

The following are equivalent:

- $R \in \Delta^0_2$,
- R is recursive with recursively enumerable oracle,
- R is of degree $\leq 0'$.

FM–truth definitions

Definition

A formula $\varphi(x)$ is an FM–truth definition in FM–domain $\mathrm{FM}(\mathcal{A})$, when for any sentence in language of $\mathcal{A}$ the following holds:
$\mathrm{FM}(\mathcal{A}) \models_{sl} \psi \equiv \varphi(\ulcorner \psi \urcorner)$.

Name(x) i Subst(x, y)

Remark

The following functions are Δ_0^0–definable:

- $\mathrm{Name}(x) = \ulcorner \underbrace{s(\dots s(}_{x} 0 \underbrace{)\dots)}_{x} \urcorner$,
- $\mathrm{Subst}_0(\ulcorner \varphi(v_0) \urcorner, \ulcorner t \urcorner) = \ulcorner \varphi(t) \urcorner$.

Corollary

Relations $\mathrm{Name}(x) = y$ and $\mathrm{Subst}_0(x, y) = z$ are FM–representable in $\mathrm{FM}(\mathcal{N})$.

Diagonal lemma

Theorem (M. Mostowski)

For every arithmetical formula $\varphi(x)$ there is an arithmetical sentence ψ such that, $\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \equiv \varphi(\ulcorner \psi \urcorner)$.

Undefinability of truth

Theorem (M. Mostowski)

(FM–version of Tarski's undefinability of truth theorem)
There is no such arithmetic formula $\varphi(x)$ that for every arithmetic sentence ψ:
$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \equiv \varphi(\ulcorner \psi \urcorner)$.

Contents

Poor arithmetics in standard model

Theorem

The first order theory of arithmetic of multiplication is decidable.

Remark

- $x|y \equiv_{df} \exists z\, x \times z = y$,
- $x \perp y \equiv_{df} \forall z\, ((z|x \wedge z|y)) \rightarrow \forall w\, z|w)$

Corollary

The first order theory of arithmetic of divisibility is decidable.

Corollary

The first order theory of arithmetic of coprimality is decidable.

Remark

In standard model with multiplication only $<$ is not definabe.

Poor arithmetics in finite models

Remark

The following formula defines order on an initial segment of any finite model of arithmetic of multiplication:

$$\varphi^{\times}_{<}(x, y) \equiv_{df} \exists c(\exists d\, x \times c = d \wedge \neg \exists e\, y \times c = e)$$

Remark

The following formula defines order on an innitial segment of any finite model of arithmetic of divisibility:

$$\varphi^{|}_{<}(x, y) \equiv_{df} \exists z\, (z \perp x \wedge z \perp y \wedge \exists w\, (x | w \wedge z | w) \wedge \neg \exists w\, (y | w \wedge z | w))$$

Problem:

Is it possible to have such an order in language of coprimality in finite models?

Contents

Definition

Let $a \in \omega$. By $\mathrm{Supp}(a)$ we denote a set of prime numbers dividing a. For $a, b \in \omega$ we write $a \approx b$ when $\mathrm{Supp}(a) = \mathrm{Supp}(b)$.

Remark

In language of coprimality numbers which are divisible by exactly the same prime numbers are indistinguishable. Thus $\approx$ is a congruence relation.

Remark

The following formula defines the relation $\approx$:

$$\varphi_{\approx}(x, y) \equiv_{df} \forall z (x \perp z \equiv y \perp z)$$

Notation

By p_i we denote the i–th prime number. $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, $p_3 = 7, \ldots$

First order interpretation of $\mathcal{A}$ in $\mathcal{B}$

Definition

Let σ and τ be relational languages. Let σ contains predicates $R_1, \ldots, R_k$ of arities $s_1, \ldots, s_k$. A sequence $\overline{\varphi} = (\varphi_U, \varphi_\approx, \varphi_{R_1}, \ldots, \varphi_{R_k})$ of formulae in language of τ is a first order interpretation of models in language of σ, when the free variable of φ_U is x_1, free variables of $\varphi_\approx$ are x_1 and x_2 and the free variables of formulae φ_{R_i} are $x_1, \ldots, x_{s_i}$ for $i = 1, \ldots, k$. The sequence $\overline{\varphi}$ defines a model in language σ in a model in language τ in the following sense:

- The universe U is defined by φ_U: $U = \{a : \mathcal{A} \models \varphi_U[a]\}$,
- The relation $\approx$ is given by the formula $\varphi_\approx$ which have to define a congruence on U,
- For $i = 1, \ldots, k$ interpretations of R_i are given by $\mathbf{R}_i(\mathbf{a}_1, \ldots, \mathbf{a}_{s_i})$ iff:
  - $\exists a_1 \in \mathbf{a}_1 \ldots \exists a_{s_i} \in \mathbf{a}_{s_i} \mathcal{A} \models \varphi_{R_i}[a_1, \ldots, a_{s_i}]$, where $\mathbf{a}_1, \ldots, \mathbf{a}_{s_i}$ are equivalence classes of relation defined on U by $\varphi_\approx$.

Interpretation of an FM–domain in another FM–domain

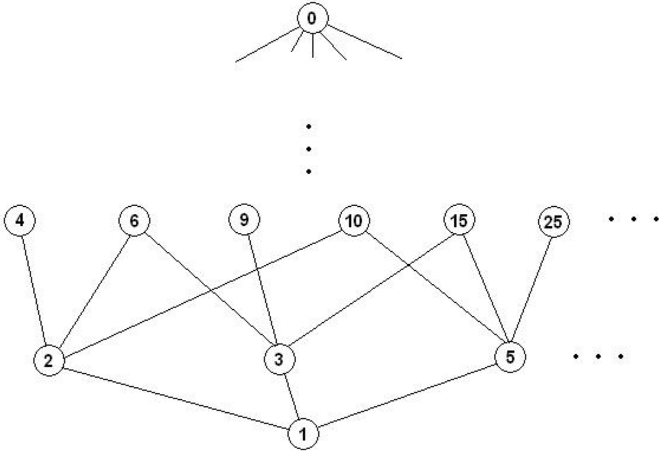
Definition

We say that $\overline{\varphi}$ is a first order interpretation of $\mathrm{FM}(\mathcal{A})$ in $\mathrm{FM}(\mathcal{B})$ if there is a monotone unbound function $f : \omega \longrightarrow \omega$ and $k \in \omega$, such that for all $n \geq k$:

$$I_{\overline{\varphi}}(\mathcal{B}_n) \cong \mathcal{A}_{f(n)},$$

where $I_{\overline{\varphi}}(\mathcal{B}_n)$ is the model defined in $\mathcal{B}_n$ by the sequence of formulas $\overline{\varphi}$.

The lattice of divisibility



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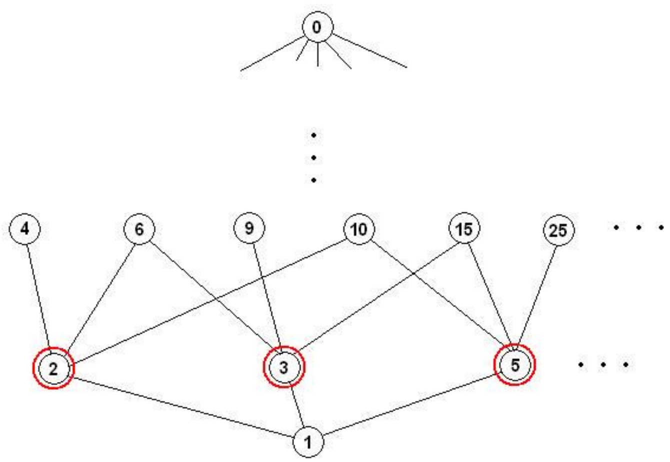

An ordering on coprimality models

Remark

Basic definitions:

- $P(x) \equiv_{df} \forall y \forall z((\neg z \perp x \wedge \neg y \perp x) \to \neg z \perp y)$ – x is a power of prime number,
- $\varphi_{\cup}(x, y, z) \equiv_{df} \forall w(w \perp z \equiv (w \perp x \wedge w \perp y))$ – $\mathrm{Supp}(z) = \mathrm{Supp}(x) \cup \mathrm{Supp}(y)$,
- $\varphi_{\prec}(x, y) \equiv_{df}$ $\exists z(P(z) \wedge x \perp z \wedge y \perp z \wedge \exists w \varphi_{\cup}(x, z, w) \wedge \neg \exists w \varphi_{\cup}(y, z, w))$ – Relation defining ordering on classes of (small) prime numbers.

Theorem (M. Mostowski, K. Zdanowski)

There is a first order interpretation of FM–domain $\mathrm{FM}(\mathcal{N})$ in the FM–domain $\mathrm{FM}((\omega, \perp))$.

Remark

The interpretation is made on the equivalence classes of the relation $\approx$ represented by prime numbers. In language of coprimality – in finite models – there are definable relations $R_+(x, y, z), R_\times(x, y, z)$ such that:

- $R_+(x, y, z) \equiv x \approx p_i \wedge y \approx p_j \wedge z \approx p_k \wedge k = i + j$,
- $R_\times(x, y, z) \equiv x \approx p_i \wedge y \approx p_j \wedge z \approx p_k \wedge k = i \times j$.

Existence of interpretation and existence of translation

Definition

Let $R \subseteq \omega^r$ be an arithmetical formula. The translation of R to the indices of prime numbers is the relation R^* such that:

$$R^*(x_1, \ldots, x_r) \equiv \exists a_1 \ldots \exists a_r (\bigwedge_{i=1}^{r} (x_i \approx p_{a_i}) \wedge R(a_1, \ldots, a_r))$$

Theorem

A relation $R \subseteq \omega^r$ is FM–representable in $\mathrm{FM}(\mathcal{N})$ iff its translation R^* is FM–representable in $\mathrm{FM}((\omega, \perp))$.

Remark

We denote the translation of $\ulcorner\ \urcorner$ by GN.

Sample translations

Examples

1.
   - $R = \{1, 5, 6\}$
   - $R^* = \bigcup_{k>0}\{p_1^k, p_5^k, p_6^k\} = \bigcup_{k>0}\{3^k, 13^k, 17^k\}$
2.
   - $S = \{(3, 4), (2, 5), (1, 1)\}$
   - $S^* = \bigcup_{k,l>0}\{(p_3^k, p_4^l), (p_2^k, p_5^l), (p_1^k, p_1^l)\} = \bigcup_{k,l>0}\{(7^k, 11^l), (5^k, 13^l), (3^k, 3^l)\}$

Diagonal lemma for $\mathrm{FM}((\omega, \perp))$

Theorem

For every formula in language of coprimality $\varphi(x)$ there exists a sentence ψ in language of coprimality such that:

$$\mathrm{FM}((\omega, \perp)) \models_{sl} \psi \equiv \varphi(\mathrm{GN}(\psi))$$

Proof of the diagonal lemma for $\mathrm{FM}((\omega, \perp))$

Let $\mathrm{Name}^*(x) \approx y$ and $\mathrm{Subst}^*(x, y) \approx z$ be translations of relations FM–representing $\mathrm{Name}(x) = y$ and $\mathrm{Subst}(x, y) = z$ respectively in $\mathrm{FM}(\mathcal{N})$. Then the following conditions hold:

- $\mathrm{FM}((\omega, \perp)) \models_{sl} \mathrm{Name}^*(x) \approx \mathrm{GN}(\underbrace{s(\ldots s(}_{x} 0 \underbrace{) \ldots)}_{x})$,
- $\mathrm{FM}((\omega, \perp)) \models_{sl} \mathrm{Subst}^*(\mathrm{GN}(\varphi(x)), \mathrm{GN}(t)) \approx \mathrm{GN}(\varphi(t))$.

Let us fix a formula $\varphi(x)$ with one free variable x. Auxiliary definitions:

- $\zeta(x) \equiv_{df} \varphi(\mathrm{Subst}^*(x, \mathrm{Name}^*(x)))$,

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Proof of the diagonal lemma

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- $\varphi(\mathrm{GN}(\zeta(m)))$,
- $\varphi(\mathrm{GN}(\psi))$.

Undefinability of truth

Undefinability of truth in $\mathrm{FM}((\omega, \perp))$

There is no formula $\varphi(x)$ in language of coprimality such that for every sentence ψ in language of coprimality holds:

$$\mathrm{FM}((\omega, \perp)) \models_{sl} \psi \equiv \varphi(\mathrm{GN}(\psi))$$

Proof:

Suppose there is such a formula $\varphi(x)$. By the diagonal lemma used to the formula $\neg\varphi(x)$ – we get a sentence ψ_0 such that:

- $\mathrm{FM}((\omega, \perp)) \models_{sl} \psi_0 \equiv \neg\varphi(\mathrm{GN}(\psi_0))$,

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- $\mathrm{FM}((\omega, \perp)) \models_{sl} \psi_0 \equiv \neg\psi_0$.

Contradiction. There is no FM–truth definition $\varphi(x)$ for $\mathrm{FM}((\omega, \perp))$.

Summary

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Summary

- There is a first order interpretation of $\mathrm{FM}(\mathcal{N})$ in $\mathrm{FM}((\omega, \perp))$,
- Coprimality is sufficient to express semantics in finite models,
- Truth is not definable in finite models of coprimality,
- The first order theory of sentences true in $\mathrm{FM}((\omega, \perp))$ is undecidable.

M. Mostowski.
On Representing Concepts in Finite Models.
Mathematical Logic Quarterly, 47:513–523, 2001.

M. Krynicki, K. Zdanowski.
Theories of Arithmetics in Finite Models.
The Journal of Symbolic Logic, 70(1):1–28, 2005.

M. Mostowski, A.E. Wasilewska.
Arithmetic of Divisibility in Finite Models.
Mathematical Logic Quarterly 50(2):169–174, 2004.

M. Mostowski, K. Zdanowski.
Coprimality in Finite Models.
Computer Science Logic: 19th Intrnational Workshop, CSL 2005, vol. 3634 of Lecture Notes in Computer Science, str. 263–275, Springer 2005.

Thank you for your attention