1. What is wrong with the following “proof” that the language \{0^n1^n : n \geq 0\} is regular?
   (a) The language \{0^n : n \geq 0\} is regular, because there is a simple DFA accepting it.
   (b) The language \{1^n : n \geq 0\} is regular, because there is a simple DFA accepting it.
   (c) Regular languages are closed under concatenation.
   (d) The language \{0^n : n \geq 0\} \cdot \{1^n : n \geq 0\} is regular (where “\cdot” denotes concatenation).
   (e) The language in (d) is \{0^n1^n : n \geq 0\} by definition of concatenation.

   Hence, \{0^n1^n : n \geq 0\} is regular.

   **Solution:** Statements (a) through (d) are true, but statement (e) is false. The languages in (d) and (e) are different.

   In (d), the definition of each set uses a variable named \(n\), but the two variables aren’t the same. Each \(n\) is local to the definition of its set. So we could have written the language equally well with two different variables, e.g. \{0^p : p \geq 0\} \cdot \{1^m : m \geq 0\}.

   This means that the language contains any string of zeros followed by any string of ones, but the number of zeros doesn’t have to match the number of ones.

   In (e), there is only one variable \(n\). So the number of zeros in the strings has to match the number of ones.

2. Assuming only that the language \{0^n1^n : n \geq 0\} is not regular, prove using closure properties that the language \{0^i1^j : i \neq j\} is not regular.

   **Solution:** Suppose that \(L = \{0^i1^j : i \neq j\}\) were regular. We know that \(M = \{0^i1^j\}\) is regular, because it has the regular expression \(0^*1^*\). Then \(M - L\) must be regular, because we know that regular expressions are closed under set difference. But \(M - L = \{0^n1^n : n \geq 0\}\), which we know not to be regular. This is a contradiction.

   Our assumption that \(L = \{0^i1^j : i \neq j\}\) was regular led to a contradiction. So, \(L = \{0^i1^j : i \neq j\}\) must not be regular.
Another possible proof is to intersect the complement with $0^*1^*$, which gives \{0^n1^n : n \geq 0\}.

3. Argue that the language \{wxw^R : w, x \in \{0, 1\}^+\} is regular, where \{0, 1\}^+ denotes one or more 0s or 1s.

**Solution:** Yup. It does look like a non-regular language, because the definition contains two arbitrarily-long parts that look like they need to match. However, we can write a different (but equivalent) definition for the same language, without needing long matching substrings.

Consider a string \(wxw^R\) from the language. Notice that \(w\) and \(x\) can be any strings of zeros and ones, so long as they contain at least one character. If \(w\) is longer than a single character, we can rewrite \(wxw^R\) using a shorter \(w\) and a longer \(x\). Specifically, suppose that \(w = va\) and \(y = axa\), where \(a\) is a single character. Then \(wxw^R = vaxav^R = vyv^R\). If we keep doing this, we can reduce the outer string to a single character. Thus, the language definition can be rewritten as \{\(wxw^R : w, x \in \{0, 1\}^+\) and \(|w| = 1\}\}. That is, the language actually contains all strings of zeros and ones whose first and last characters match. It’s easy to show that this language is regular, e.g. it is the language of the regular expression \((0(0 + 1)^*0) + (1(0 + 1)^*1))\).

4. Prove using the pumping lemma that the set of palindromes over \{0, 1\} is not regular.

**Solution:** Suppose that the set of palindromes were regular. Let \(n\) be the value from the pumping lemma. Consider the string \(w = 0^n110^n\). \(w\) is clearly a palindrome and \(|w| \geq n\). By the pumping lemma, there must exist strings \(x, y,\) and \(z\) satisfying the four constraints of the pumping lemma.

So, pick any \(x, y,\) and \(z\) such that \(w = xyz, |xy| \leq n,\) and \(|y| \geq 1\). Because \(|xy| \leq n, xy\) is entirely contained in the \(0^n\) at the start of \(w\). So \(x\) and \(y\) consist entirely of zeros, i.e. \(x = 0^i\) and \(y = 0^j\). Then \(z\) must look like \(0^k110^n\), where \(i + j + k = n\).

Now, consider \(xz\). By the pumping lemma, \(xz\) must be in the language. But \(xz = 0^i \cdot 0^k110^n\). This is just \(0^{i+k}110^n\). Since \(|y| \geq 1\), we know that \(j \geq 1\). So \(i + k < n\). This means that \(xz\) is not a palindrome, because the numbers of zeros on the two ends don’t match.

This means that the set of palindromes doesn’t satisfy the pumping lemma and, thus, the set of palindromes cannot be regular.

5. Let
\[
\Sigma_3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]
\(\Sigma_3\) contains all size 3 columns of 0s and 1s. A string of symbols in \(\Sigma_3\) gives three rows of 0s and 1s. Consider each row to be a binary number and let
\[L = \{ w \in \Sigma_3 : \text{the bottom row of } w \text{ is the product of the top two rows}\}.\]
Use the pumping lemma to show that $L$ is not regular.

**Solution:** Suppose that $L$ were regular. Let $n$ be the value from the pumping lemma. Consider the string:

$$w = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)^n \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^n$$

That is, we are multiplying $0^n10^n$ (i.e. $2^n$) by $0^n1^{n+1}$ (i.e. $2^{n+1} - 1$), yielding the correct product $1^{n+1}0^n$ (i.e. $2^{2n+1} - 2^n$).

Since $|w| \geq n$, there must exist strings $x$, $y$, and $z$ satisfying the four constraints of the pumping lemma. Thus, $x$, $y$, and $z$ satisfy $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$.

The pumping lemma says that $xz$ must be in $L$. But what does $xz$ look like? Because $|xy| \leq n$, $x$ and $y$ consist entirely of triples $\left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right)$. Suppose that $|y|$ is $j$. Then $xz$ looks like

$$xz = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)^{n-j} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^n$$

The top two rows of $xw$ are exactly the same as in $w$, because we are removing only leading zeros. However, because $j \geq 1$, we have removed some leading ones from the product in the bottom row. So the bottom row is no longer the product of the top two rows. So $xz$ is not in $L$, contradicting the claim from the pumping lemma.

So $L$ doesn’t satisfy the pumping lemma and, therefore, $L$ cannot be regular.