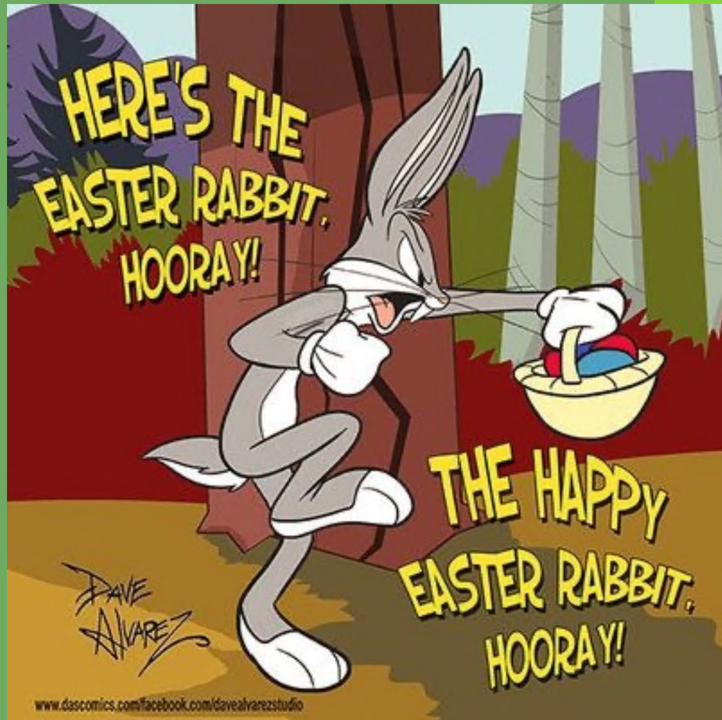
PDEs I: Tutorial 3

18.03.2021



Problem B6 / PS2

Znaleźć wszystkie $u \geq 0$ t. że

$$\begin{cases} \Delta u = u^2 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

Rozw: $\Delta u \geq 0 \Rightarrow u$ jest podharm.

$$\Rightarrow \begin{cases} u \leq 0 \\ u \geq 0 \end{cases} \Rightarrow \underline{u = 0}. \longrightarrow \text{to jest rozw.}$$

Problem 12 / PS 1

$$\partial_t u(t,x) + b(x)\,\partial_x u(t,x) = 0$$

funkcja różniczk.

$\hookrightarrow$ rozwiązanie na charakt.

funkcje

$\longrightarrow$ $\underbrace{\text{słabe (dystr.) rozwiązania}}$

pozwala rozważać nieróżniczk. rozw.

miary

$\longrightarrow$ rozwiązanie w przestrzeni miar

δ_{x_0} = skoncentrowane

$\approx \delta_{x_0}$

definicja: $\{\mu_t\}_{t\in[0,T]}$ jest rozw. miarowym jeżeli

$\forall_{\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})}$ mamy

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t \varphi(t,x)\, \underbrace{d\mu_t(x)}\, \underbrace{dt} + \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x\big(b(x)\varphi(t,x)\big)\, d\mu_t(x)\, dt$$

$$+ \int_{\mathbb{R}} \varphi(0,x)\, \underbrace{d\mu_0}(x) = 0,$$

warunek początkowy

(A) Zał. że $\{\mu_t\}$ jest rozw. miarowym i μ_t ma gęstość względem miary Lebesguea $u(t,x)$ która jest C^1.

Pokażemy że $u(t,x)$ spełnia równanie transportu.

$$\mu_t(A) = \int_A u(t,x)\,dx$$

(AM II.1:
→ przez funkcje proste...)

$$\int_{\mathbb{R}} f(x)\,d\mu_t(x) = \int_{\mathbb{R}} f(x)\,u(t,x)\,dx$$

Założeniowy:

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t \varphi(t,x)\, d\mu_t(x)\, dt + \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x \big(b(x)\varphi(t,x)\big)\, d\mu_t(x)\, dt$$

$$+ \int_{\mathbb{R}} \varphi(0,x)\, d\mu_0(x) = 0.$$

Ponieważ μ_t ma gęstość $u(t,x)$ to

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t \varphi(t,x)\, u(t,x)\, dx\, dt + \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x \big(b(x)\varphi(t,x)\big)\, u(t,x)\, dx\, dt$$

$$+ \int_{\mathbb{R}} \varphi(0,x)\, u(0,x)\, dx = 0.$$

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t \varphi(t,x)\, u(t,x)\, dx\, dt + \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x \big(b(x)\, \varphi(t,x)\big)\, u(t,x)\, dx\, dt$$

$$+ \int_{\mathbb{R}} \varphi(0,x)\, u(0,x)\, dx = 0.$$

0 bo φ ma zwarty nośnik

$$-\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)\, \partial_t u(t,x)\, dx\, dt + \int \varphi(t,x)\, u(t,x)\, dx \Big|_{t=0}^{t=\infty}$$

$$= -\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)\, \partial_t u(t,x)\, dx\, dt - \int_{\mathbb{R}} \varphi(0,x)\, u(0,x)\, dx$$

$$-\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)\, b(x)\, \partial_x u(t,x)\, dx\, dt$$

$$\int \varphi(t,x)\left[b(t,x)\,\partial_x u(t,x) + \partial_t u(t,x)\right] = 0$$

$$\forall_{\varphi} \quad \Rightarrow \quad b(t,x)\,\partial_x u(t,x) + \partial_t u(t,x) = 0$$

(B) Dla $b(x) = b \in \mathbb{R}$, $\mu_0 = \delta_{x_0}$

wyznacz $\mu_t = \delta_{x_0 + tb}$ (propozycje)

$$\mu_t = \delta_{x_0 + tb} \quad (\text{propozycja})$$

musimy sprawdzić

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t \varphi(t,x)\, \underbrace{d\mu_t(x)}\, \underbrace{dt} + \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x \big(b(x)\varphi(t,x)\big)\, d\mu_t(x)\, dt + \int_{\mathbb{R}} \varphi(0,x)\, \underbrace{d\mu_0(x)} = 0.$$

$$\int_{\mathbb{R}^+} \partial_t \varphi(t, x_0 + tb)\, dt + \int_{\mathbb{R}^+} b\, \partial_x\big(\varphi(t, x_0+tb)\big)\, dt + \varphi(0, x_0).$$

$$\int_{\mathbb{R}^+} \frac{d}{dt} \varphi(t, x_0 + tb)\, dt + \varphi(0, x_0) = 0$$

$$\varphi(t, x_0 + tb)\Big|_{t=0}^{t=\infty} + \varphi(0, x_0) = 0.$$

$$\parallel$$

$$0 - \varphi(0, x_0) + \varphi(0, x_0) = 0$$

"Fajnie jest!"

<u>HW</u>: $\begin{cases} \partial_t \mu_t + \partial_x (b(x)\mu_t) = 0 \\ \mu_t\big|_{t=0} = \mu_0 \in \mathcal{M}^+ \end{cases}$

push-forward !!! $\longrightarrow$ optymalny transport

równania hiperboliczne (r-nie transportu)

1) rozwiązania propagują się na krzywych.

2) pochodne rzędu ≤ 1.

3) regularność rozw. taka sama jak war. początkowego lub gorsza (nieliniowe problemy)

4) $\partial_t u(t,x) + b(x)\,\partial_x u(t,x) = 0$

$$u(t,x) = u_0\left(X_b(-t,x)\right)$$

tw. o dyw.

$$\boxed{\int_{\Omega} \operatorname{div} F \, dx = \int_{\partial\Omega} \langle F, \vec{n} \rangle \, dS(x)}.$$

$F: \mathbb{R}^n \to \mathbb{R}^n$

<u>A2</u>:

$$\int_{\Omega} \overbrace{v}^{0} \, \overbrace{\Delta u}^{2} + \int_{\Omega} \overbrace{\nabla u}^{1} \cdot \overbrace{\nabla v}^{1} = \int_{\partial\Omega} v \, \frac{\partial u}{\partial n}$$

<u>D-d</u>:

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS(x) \qquad (\text{dla } F = \nabla u)$$

$$F = v \cdot \nabla u$$

$$\int_{\Omega} \operatorname{div} F = \int_{\partial\Omega} \langle F, n \rangle$$

$$\operatorname{div}(v \cdot \nabla u) = \sum_{i=1}^{n} \partial_{x_i}(v\, \partial_{x_i} u) = \sum_{i=1}^{n} v \cdot \partial^2_{x_i} u$$

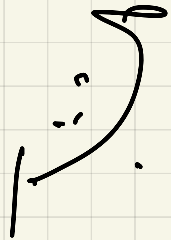
$$+ \sum_{i=1}^{n} \partial_{x_i} v \cdot \partial_{x_i} u = v \Delta u + \nabla v \cdot \nabla u$$

$$\langle F, n \rangle = v \cdot \frac{\partial u}{\partial n}$$

$$\int_{\Omega} v \Delta u + \int_{\Omega} \nabla v \cdot \nabla u = \int_{\partial\Omega} v \cdot \frac{\partial u}{\partial n}$$

$$\boxed{A3} \quad \int_{\Omega} \left(v \Delta u - \Delta v\, u \right) = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right)$$

wynika z A2 po odjęciu stronami



$\boxed{A4}$

$$\int_\Omega \partial_j u \cdot v + \int_\Omega u \cdot \partial_j v = \int_{\partial\Omega} u(x)\, v(x)\, n_j(x)\, dS(x)$$

$$\int_\Omega \operatorname{div} F = \int_{\partial\Omega} \langle F, n \rangle \qquad n = (n_1, n_2, \dots, n_n)$$

$$F = (0, 0, \dots, 0, \underbrace{u \cdot v}_{j\text{-ta}}, 0, \dots, 0)$$

$$\operatorname{div} F = \partial_j (u \cdot v) = \qquad \Big| \quad \langle F, n \rangle =$$

$$= \partial_j u \cdot v + u \cdot \partial_j v. \qquad \Big| \quad u \cdot v \cdot n_j$$

Zasada maksimum dla funkcji pod/nad harm.:

$$\begin{cases} -\Delta u_1 = f_1 & \Omega \\ u_1 = g_1 & \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u_2 = f_2 & \Omega \\ u_2 = g_2 & \partial\Omega \end{cases}$$

1) $f_1 \leq f_2, \; g_1 \leq g_2 \Rightarrow u_1 \leq u_2$

2) $\|u_1\|_\infty \leq C\left[\|f_1\|_\infty + \|g_1\|_\infty\right]$

3) $\|u_1 - u_2\|_\infty \leq C\left[\|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty\right]$

Problem C1:

jednoznaczność dla $\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$ (*)

u_1, u_2 spełniają (*) z takimi samymi f i g

$\Rightarrow u_1 = u_2$.

1) $\begin{cases} -\Delta(u_1 - u_2) = 0 & \Omega \\ u_1 - u_2 = 0 & \partial\Omega \end{cases} \Rightarrow$ $u_1 - u_2$ jest harm.

$\Downarrow$

$u_1 - u_2 = 0 \ \Omega$

2) stabilność: $\|u_1 - u_2\|_\infty = 0$.

Zadanie C2:

$$\begin{cases} -\Delta u_1 = f \\ u_1 = g \end{cases} \qquad \begin{cases} -\Delta u_2 = f \\ u_2 = g \end{cases}$$

$u_1 - u_2$

$$\begin{cases} -\Delta(u_1-u_2) = 0 & \Omega \\ u_1 - u_2 = 0 & \partial\Omega \end{cases} \qquad / \cdot (u_1-u_2)$$

$$(u_1-u_2)\,\Delta(u_1-u_2) = 0 \text{ na } \Omega \qquad / \int_\Omega$$

$$\int_\Omega \nabla(u_1-u_2)\,\nabla(u_1-u_2) = 0$$

$$\|$$

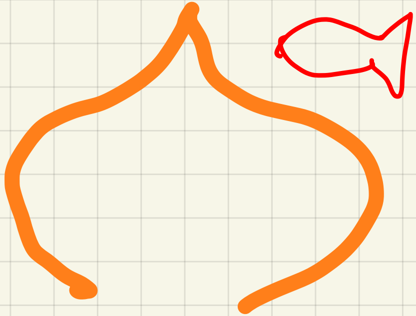
$$\int_\Omega v\,\Delta u + \int_\Omega \nabla v \cdot \nabla u = \underbrace{\int_{\partial\Omega} v \cdot \frac{\partial u}{\partial n}}$$

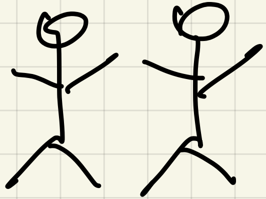
$$0 = \int_\Omega (u_1-u_2)\,\Delta(u_1-u_2) = -\int_\Omega |\nabla(u_1-u_2)|^2$$

$$+ \int_{\partial\Omega} \underbrace{(u_1-u_2)}_{=\,0 \text{ na } \partial\Omega} \frac{\partial(u_1-u_2)}{\partial n} = -\int_\Omega |\nabla(u_1-u_2)|^2$$

$\Rightarrow$ $\nabla(u_1-u_2) = 0$ Ω $\Rightarrow$ $\boxed{u_1-u_2 = 0.}$

$u_1-u_2 = 0$ na $\partial\Omega$





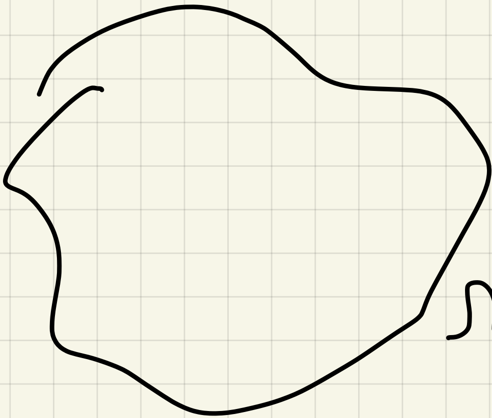
$$-\Delta u = f \quad \Omega$$
$$u = g \quad \partial\Omega$$

$$\Omega = B_r(0)$$

„wzór Poisson"*

CEL:

explicite wzór na u dla $\Omega = B_r(0)$.



Ω

Problem D1: (PS2)

$$\Phi(x) = \begin{cases} \frac{1}{n(2-n)\alpha_n} |x|^{2-n} & n \geq 3 \\ -\log|x| & n = 2 \end{cases}$$

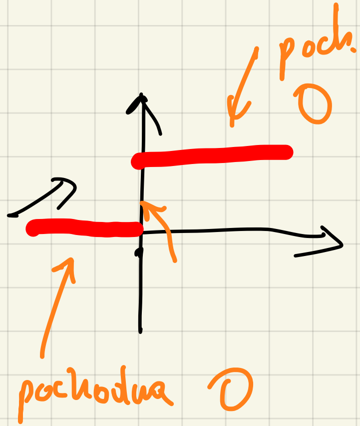
rozw. fundamentalne dla $\Delta u = 0$:

rozw. radialnie symetryczne które spełnia $\Delta u = 0$ dla $x \neq 0$.

$$-\Delta u(x) = \delta_0$$

1) miara skoncentrowana w 0

2) "funkcja" która ma skok w 0 $\Rightarrow$



3) (poł. z 1) funkcjonał liniowy na $C(\mathbb{R}^n)$ t. że $\varphi(f) = f(0)$.

narazie intuicja, za 2-3 zajęcia nabierze precyzji.

Problem D2 rozw. fund. w 1D

$$u''(x) = \delta_0(x)$$

u jest symetryczne $u(x) = u(-x)$.

$$u''(x) = 0$$

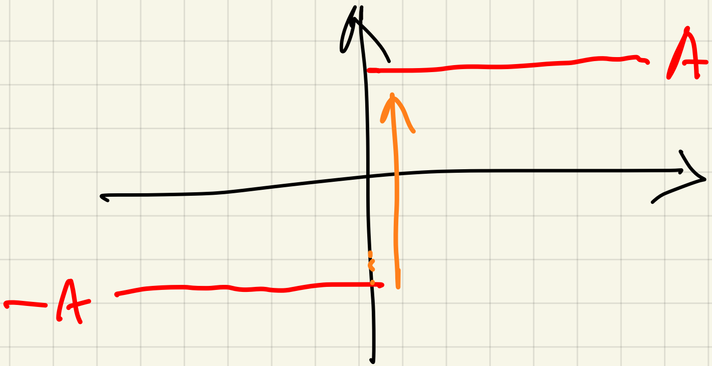
$u(x) = Ax + B$ $\exists_{A,B}$ $B = 0$

$u(x) = Ax$ $x > 0$

$$u(x) = A|x|$$

$$u(x) = A|x| \qquad\qquad u''(x) = \delta_0(x)$$

$$u'(x) = \begin{cases} -A & x < 0 \\ A & x > 0 \end{cases}$$



$$\underbrace{\delta_0 \cdot (2A)} = \delta_0$$

$$A = \frac{1}{2}.$$

$$\boxed{u''(x) = \delta_0(x) \qquad u(x) = \frac{1}{2}|x|.}$$

Zadanie D3: jeżeli $u \in C^2(\bar{\Omega})$

$$u(x) = -\int_{\partial\Omega} u(y)\, \frac{\partial \Phi}{\partial n}(y-x)\, dS(y)$$

$$+ \int_{\partial\Omega} \Phi(y-x)\, \frac{\partial u}{\partial n}(y)\, dS(y)$$

$$- \int_{\Omega} \Phi(y-x)\, \Delta u(y)\, dy.$$

Wtorek: formalny dowód ze wszystkimi detalami.

$$\int_\Omega \left(v\,\Delta u - \Delta v\, u \right) = \underbrace{\int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right)}_{\text{(to było)}}$$

(to chcemy)

$$u(x) = -\int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial n}(y-x)\, dS(y)$$

$$+ \int_{\partial\Omega} \Phi(y-x) \frac{\partial u}{\partial n}(y)\, dS(y)$$

$$- \int_{\Omega} \Phi(y-x)\, \Delta u(y)\, dy.$$

$u = u(y)$

$v = \Phi(y-x)$

x – ustalonym punktem

$$\int_\Omega \underbrace{(v\,\Delta u}_{} - \Delta v\, u) = \underbrace{\int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)}_{}$$

(to było)

$$\|$$

$$\int_\Omega \Phi(y-x)\,\Delta u(y) - \int_\Omega u(y)\,\Delta\Phi(y-x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial n}(y) - \int_{\partial\Omega} u(y)\frac{\partial\Phi(y-x)}{\partial n}$$

$$\text{''} \quad -$$

tego wzoru nie można zast. tego wzoru ale ... jednak można.

$$v(y) = \Phi(y-x) \qquad u(y) = u(y)$$

$$-\int_{\Omega} u(y)\, \Delta \Phi(y-x)\, dy$$

nieformalnie
$$\Delta \Phi(y) = -\delta_0(y)$$
$$\Delta \Phi(y-x) = -\delta_x(y)$$

$$+\int_{\Omega} u(y)\, \delta_x(y) = u(x).$$

D4

$$u(x) = -\boxed{\int_{\partial\Omega} u(y)} \frac{\partial \Phi}{\partial n}(y-x)\, dS(y)$$

$$+ \int_{\partial\Omega} \Phi(y-x) \boxed{\frac{\partial u}{\partial n}(y)}\, dS(y)$$

$$- \int_{\Omega} \Phi(y-x) \boxed{\Delta u(y)}\, dy.$$

Jak to zastosować:

$$-\Delta u = f \quad \Omega$$

$$u = g \quad \partial\Omega$$

$$u(x) = -\boxed{\int_{\partial\Omega} u(y)} \frac{\partial \Phi}{\partial n}(y-x)\, dS(y)$$

$$- \int_{\Omega} \Phi(y-x) \boxed{\Delta u(y)}\, dy.$$

(bez 1 kawałka!)

$$u(x) = -\int_{\partial\Omega} g \frac{\partial \Phi}{\partial n}(y-x)\, dS(y) + \int_{\Omega} \Phi(y-x) f(y)\, dy$$

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

$$u(x) = -\int_{\partial\Omega} \boxed{u(y)} \frac{\partial \Phi}{\partial n}(y-x)\, dS(y)$$

$$+ \int_{\partial\Omega} \Phi(y-x) \boxed{\frac{\partial u}{\partial n}(y)}\, dS(y) - \int_{\Omega} \Phi(y-x) \boxed{\Delta u(y)}\, dy.$$

φ^x (x jest ustalone)

$$\begin{cases} -\Delta \varphi^x(y) = 0 & \Omega \\ \varphi^x(y) = \Phi(x-y) & \partial\Omega \end{cases}$$

(załóżmy, że to potrafimy rozwiązać).

$$\begin{cases} -\Delta \varphi^x(y) = 0 & \Omega \\ \varphi^x(y) = \Phi(x-y) & \partial\Omega \end{cases}$$

$$\int_\Omega \left(v \cdot \Delta u - \Delta v \cdot u \right) = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \cdot \frac{\partial v}{\partial n} \right) \quad \Bigg| \quad \begin{aligned} u(y) &= u(y) \\ v(y) &= \Phi(y-x). \end{aligned}$$

$$\int_\Omega \varphi^x(y)\, \Delta u - \underbrace{\Delta \varphi^x(y)}_{=0} u = \int_{\partial\Omega} \underbrace{\varphi^x(y)}_{=\Phi(x-y)} \frac{\partial u}{\partial n} - u \frac{\partial \varphi^x}{\partial n} \quad \Bigg| \quad \begin{aligned} u(y) &= u(y) \\ v(y) &= \varphi^x(y) \end{aligned}$$

$$\int_{\Omega} \varphi^{x}(y)\,\Delta u(y) = \int_{\partial\Omega} \Phi(y-x)\,\frac{\partial u}{\partial n} - u\,\frac{\partial \varphi^{x}(y)}{\partial n}$$

$$u(x) = -\int_{\partial\Omega} u(y)\,\frac{\partial \Phi}{\partial n}(y-x)\,dS(y) + \int_{\partial\Omega} \Phi(y-x)\,\frac{\partial u}{\partial n}(y)\,dS(y) - \int_{\Omega} \Phi(y-x)\,\Delta u(y)\,d$$

$$u(x) - \int_{\Omega} \varphi^{x}(y)\,\Delta u(y) = -\int_{\partial\Omega} u(y)\,\frac{\partial \Phi}{\partial n}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\,\Delta u(y) + \int_{\partial\Omega} u(y)\,\frac{\partial \varphi^{x}(y)}{\partial n}.$$

$$u(x) - \int_{\Omega} \varphi^x(y)\, \Delta u(y) = - \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial n}(y-x)\, dS(y)$$

$$- \int_{\Omega} \Phi(y-x)\, \Delta u(y) + \int_{\partial\Omega} u(y) \frac{\partial \varphi^x(y)}{\partial n}.$$

$$u(x) = - \int_{\partial\Omega} u(y) \left[\frac{\partial \Phi(y-x)}{\partial n} - \frac{\partial \varphi^x(y)}{\partial n} \right] dS(y)$$

$$- \int_{\Omega} \Delta u(y) \left[\Phi(y-x) - \varphi^x(y) \right] dy.$$

$$G(x,y) = \Phi(y-x) - \varphi^x(y)$$

funkcja Greena.

10:05 – 10:20

spotkanie wielkanocne.

→ czekoladowy zając (jajko...)