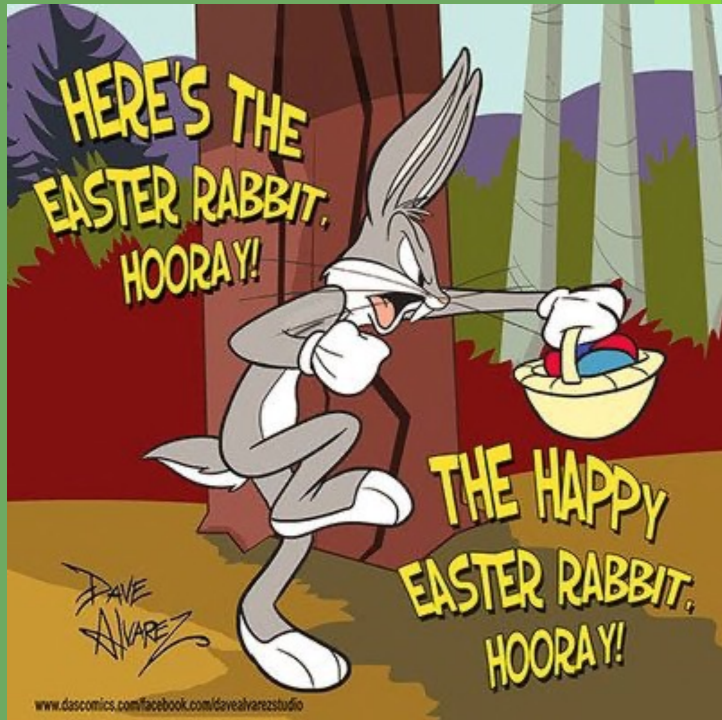
PDEs I: Tutorial 4

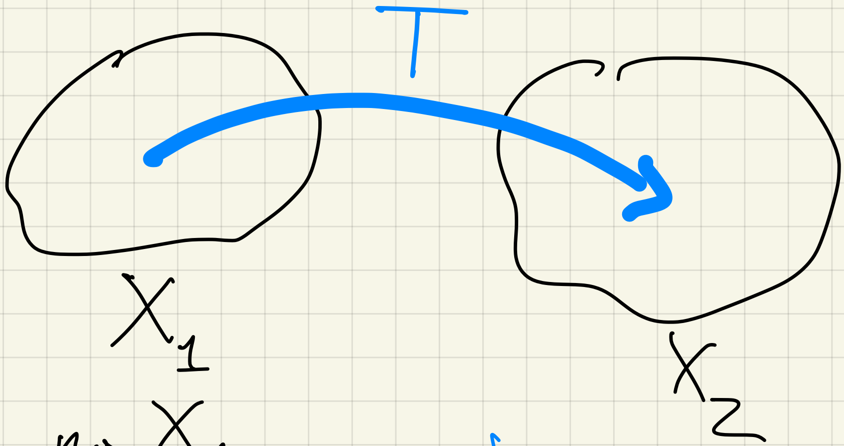
25.03.2021



Komentarz do zad. domowego (1)

push-forward μ

$T^{\#}\mu$ = miara na X_2



miara μ na X_1

tu nie ma miary

Optymalny transport:

piach

funkcja kosztu

$$C(x,y)$$
$$\|$$

koszt przes. piachu z x do y

piach ma rozkład μ ← miara.

Szef: piach był rozłożony z rozkładem ν

medale Fieldsa:

! 2010 Villani

2018 Figalli doktorant

związki optymalnego transportu z PDE.

W moodle: REVIEW (krótkie).

Zad. C4

przedyskut. jednoznaczności dla

$$-\Delta u = f \quad \text{na } \mathbb{R}^d$$

Tw o jednoznaczności: $\Omega \subset \mathbb{R}^d$ $(\Omega \approx B_r(0))$.

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

ograniczony

u_1, u_2 spełn. $-\Delta u_i = f$ na $\mathbb{R}^d$

$\Rightarrow -\Delta(u_1 - u_2) = 0 \Rightarrow u_1 - u_2$ jest harm. na $\mathbb{R}^d$

tw. Liouville'a : funkcja harmoniczna na całej przestrzeni jest stała

$$u_1 - u_2 = C = \mathrm{const}$$

Jednoznaczność na całej przestrzeni mamy z dokładnością do stałej.

RÓWNANIA
$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

- jednoznaczność
- stabilność $\rightarrow$ $\Uparrow$ $\|u\|_\infty \le C\left[\|f\|_\infty + \|g\|_\infty\right]$
- zasada maksimum, porównawcza

Brakuje: ISTNIENIE

w pop. odcinku:

Jeżeli $u \in C^2(\bar{\Omega})$ i spełnia $\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$

to

$$u(x) = \int_{\Omega} G(x,y) f(y) dy - \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n} \cdot g(y) dy.$$

$$G(x,y) = \Phi(y-x) - \varphi^x(y)$$

funkcja Greena

$$G(x,y) = \Phi(y-x) - \underbrace{\varphi^x(y)}_{\text{korektor}}$$

- $\Delta \Phi(x) = 0 \qquad x \neq 0$

- $\varphi^x(y)$ spełnia $\begin{cases} -\Delta \varphi^x(y) = 0 & \Omega \\ \varphi^x(y) = \Phi(y-x) & \partial\Omega \end{cases}$

$$\Delta_y G(x,y) = \Delta_y \Phi(y-x) - \Delta_y \underbrace{\varphi^x(y)}_{=0} \qquad x \neq y$$

suma drugich poch po y

Jeżeli $u \in C^2(\bar{\Omega})$ i spełnia $\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$

$$u(x) = \int_{\Omega} G(x,y) f(y)\, dy - \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n} \cdot g(y)\, dy.$$

- $\Delta_y G(x,y) = 0 \qquad (x \neq y)$
- (WYK) $G(x,y) = G(y,x)$
- $\Delta_x G(x,y) = 0$

(WYK) Dla kuli $\Omega = B_r(0)$ można wyznaczyć G

i $$\frac{\partial G(x,y)}{\partial n} = -\frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n}.$$

Jeżeli $u \in C^2(\overline{U})$ spełnia $\begin{cases} -\Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$

$$u(x) = \cancel{\int_{\Omega} G(x,y) f(y)\, dy} - \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n} \cdot g(y)\, dy$$

Tw (wzór Poissona)

jądro Poissona

$$u(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n} \cdot g(y)\, dy = -\int_{\partial B_r(0)} \frac{\partial G(x,y)}{\partial n} g(y) dy$$

spełnia
$$\begin{cases} -\Delta u = 0 & B_r(0) \\ u = g & \partial B_r(0) \end{cases}.$$

D-d: $\Delta u = 0$? w $B_r(0)$

$$\Delta_x G(x,y) = 0 \quad x \neq y$$

$$\Delta_x \frac{\partial}{\partial n} G(x,y) = 0 \quad x \neq y$$

$$\underline{\Delta_x \frac{\partial}{\partial n} G(x,y) = 0} \qquad x \neq y \qquad y \in \partial B_r(0)$$

$$x \in B_r(0)$$

$$u(x) = \int_{\partial B_r(0)} \underbrace{\underbrace{\frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n}}_{= \frac{\partial}{\partial n} G(x,y)} \cdot}_{\Delta_x(\dots) =} g(y)\, dy$$

$\Delta u = 0$ nie w $B_r(0)$ ale w $\underset{\displaystyle \ni x}{B_{r(1-\varepsilon)}(0)}$

wtedy $|x-y|^n \geq \varepsilon^n > 0$

Kiedy można wejść z różniczkowaniem pod całkę.

$F(x,y)$

$$x \longmapsto \int_{\Omega} F(x,y)\,dy \qquad \frac{d}{dx}\int_{\Omega} F(x,y)\,dy = \int_{\Omega} \frac{d}{dx} F(x,y)\,dy$$

$$\frac{\int_{\Omega} F(x+h,y)\,dy - \int_{\Omega} F(x,y)\,dy}{h} = \int_{\Omega} \underbrace{\frac{F(x+h,y)-F(x,y)}{h}}_{\leq \|\partial_x F\|_\infty} dy$$

skończony

$$\xrightarrow[\text{zbież.}]{\text{z tw. o zb.}} \int_{\Omega} \frac{d}{dx} F(x,y)\,dy.$$

$$\Delta u = 0 \quad \text{w} \quad B_{r(1-\varepsilon)}(0) \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \Delta u = 0 \quad \text{w} \quad B_r(0)$$

(wnętrze).

$$\Rightarrow u \in C^{\boxed{\infty}}(B_r(0))$$

(względnie: $\Delta u = 0 \Rightarrow u \in C^\infty(B_r(0))$).

(Weyl Lemma)

$$u(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n} \cdot g(y)\, dy$$

$\Delta u(x) = 0$ w $B_r(0)$ (wiemy)

$u(x) = g(x)$ na $\partial B_r(0)$.

$$\lim_{x \to x_0} u(x) = g(x_0)$$

$x_0 \in \partial B_r(0)$

$$\boxed{\begin{array}{l}\lim\limits_{x\to x_0} u(x) = g(x_0) \\ x_0 \in \partial B_r(0)\end{array}}$$

$$u(x) = \int_{\partial B_r(0)} \frac{r^2-|x|^2}{n\alpha_n r}\,\frac{1}{|x-y|^n}\cdot g(y)\,dy$$

$$u(x) - g(x_0) = \underbrace{\int_{\partial B_r(0)} \frac{r^2-|x|^2}{n\alpha_n r}\,\frac{1}{|x-y|^n}\, g(y)\,dy}_{} - \underbrace{g(x_0)}_{}$$

Chcemy $\displaystyle\int_{\partial B_r(0)} \frac{r^2-|x|^2}{n\alpha_n r}\,\frac{1}{|x-y|^n}\,dy = 1.$

(Wiemy) jak $\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$ to mamy wzór:

$$u(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n} \cdot g(y)\, dy$$

$$u = g = 1$$

$$\Rightarrow \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n}\, dy = 1.$$

$$u(x) - g(x_0) = \underbrace{\int_{\partial B_r(0)} \frac{r^2-|x|^2}{n\alpha_n r} \frac{1}{|x-y|^n} g(y)\,dy}\; - \underbrace{g(x_0)}$$

$$= \int_{\partial B_r(0)} (\ldots\ldots)\,(g(y)-g(x_0))\,dy \qquad g \in C(\partial B_r(0))$$

Ust. $\varepsilon > 0$. Istnieje δ $|x_1 - x_2| \leq \delta \Rightarrow$

$$|g(x_1) - g(x_2)| \leq \varepsilon.$$

$x \to x_0$ a $y \in \partial B_r(0)$.

$$\int_{\partial B_r(0)} (\ldots\ldots) \, \underbrace{(g(y) - g(x_0))} \, dy =$$

$$= \int_{\partial B_r(0) \cap |y - x_0| \leq \delta} (g(y) - g(x_0)) \, dy \, (\ldots) \leq \varepsilon$$

$$+ \int_{\partial B_r(0) \cap |y - x_0| > \delta} (g(y) - g(x_0)) (\ldots) \, dy$$

$$+ \int_{\partial B_r(0) \cap |y-x_0| > \delta} (g(y) - g(x_0)) (\dots) \, dy$$

$$\frac{r^2 - |x|^2}{n \alpha_n r} \frac{1}{|x-y|^n}$$

$x \to x_0$ $\qquad |x - x_0| \le \frac{\delta}{2}.$

$$|y - x_0| > \delta$$

$$\delta < |y - x_0| \le |x - x_0| + |y - x| \le \frac{\delta}{2} + |y - x|.$$

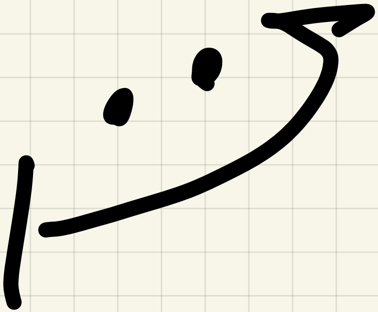
$$\Rightarrow \quad \frac{\delta}{2} \le |y - x|.$$

$$\limsup_{x \to x_0} \left| \int_{\partial B_r(0)} (\ldots)\,\underline{(g(y)-g(x_0))}\,dy \right| \leq$$

$$\varepsilon + \limsup_{x \to x_0} \int_{\partial B_r(0) \cap |y-x_0| > \delta} (g(y)-g(x_0))\,(\ldots)\,dy$$

$$\frac{r^2-|x|^2}{|x-y|^n} \leq \frac{r^2-|x|^2}{(\delta/2)^n}$$

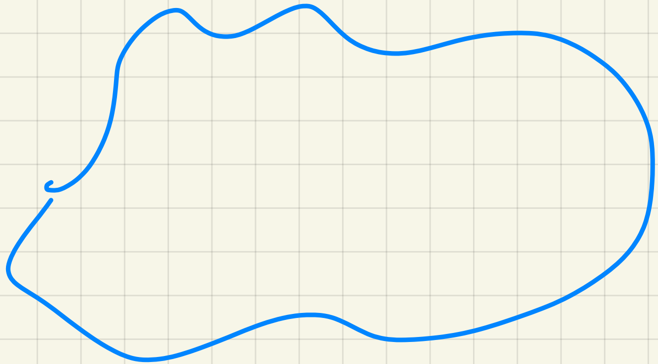
$$\leq \varepsilon$$



$$-\Delta u = 0 \qquad B_r(0)$$

$$u = g \qquad \partial B_r(0)$$

(wzór Poissona)



$$-\Delta u = 0 \qquad \underbrace{B_r(0)}$$

$$u = g \qquad \partial B_r(0)$$

Wyst. znaleźć dowolne v $\underline{-\Delta v = f}$ $B_r(0)$.

Konstrukcja
$$\begin{cases} -\Delta u = f & B_r(0) \\ u = g & \partial B_r(0) \end{cases}$$

Zał. że mam v $\quad -\Delta v = f \quad B_r(0)$

Zał. że $\forall_g$ umiem znaleźć w t. że

$$\begin{cases} -\Delta w = 0 & B_r \\ w = g - v \end{cases}$$

$u = v + w$.

$$\begin{cases} -\Delta u = f & B_r(0) \\ u = g & \partial B_r(0) \end{cases}$$

Pomysł na dowolną funkcję w_f

$$-\Delta w_f = f \qquad \Omega, \; B_r(0).$$

$$-\Delta \Phi(y-x) = 0 \qquad y \neq x$$

$$\text{"} -\Delta \Phi(y-x) = \delta_{y=x}$$

$$w_f(y) = \int_\Omega \Phi(y-x) f(x)\, dx$$

$$w_f(y) = \int_\Omega \Phi(y-x) f(x)\, dx$$

$$-\Delta w_f(y) = \int_\Omega -\Delta \Phi(y-x) f(x)\, dx$$

$$= \int f(x)\, \delta_{y=x}\, dx = f(y).$$

(motywacja)

CEL: pokazać że $-\Delta w_f = f$

$$w_f(y) = \int_\Omega \Phi(y-x) f(x)\, dx$$

$$\Phi(y-x) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x|^{2-n} & n \geq 3 \\ -\frac{1}{2\pi} \log |x| & n=2. \end{cases}$$

Problem E1:

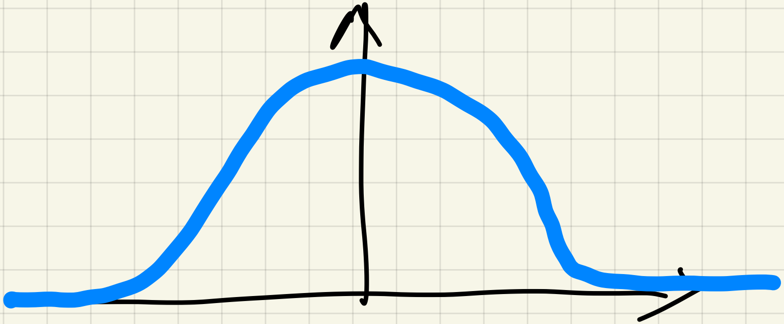
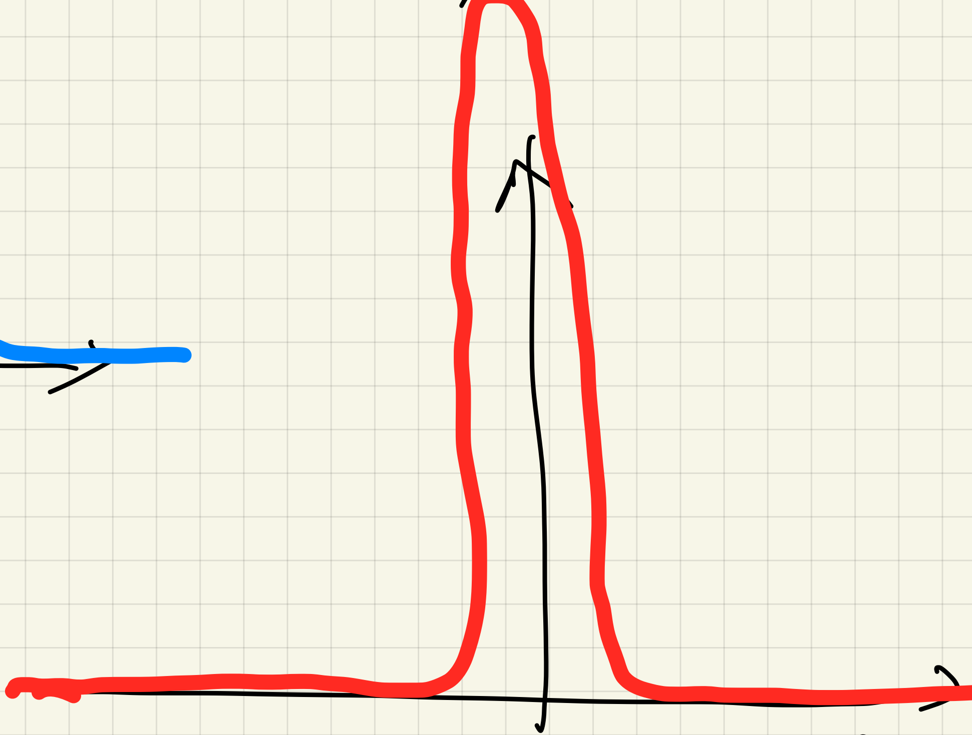
pokazać, że istnieje gładka $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$

- $\zeta(x) = 0 \qquad |x| \leq 1$
- $\zeta(x) = 1 \qquad |x| \geq 2$
- $|\zeta'(x)| \leq C \qquad \forall_{x \in \mathbb{R}^+}$

$\operatorname{supp} \eta \subset B_1(0)$

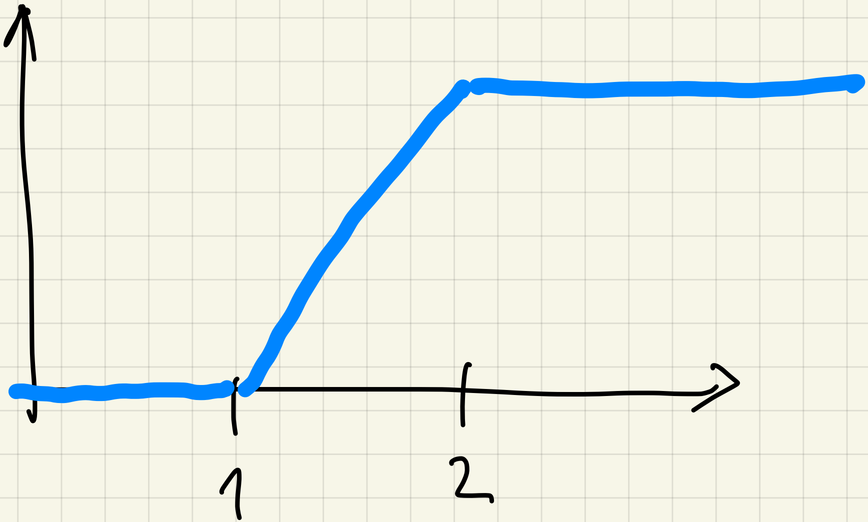
η taka, że $\eta \in C_c^\infty(\mathbb{R})$, $\eta \geq 0$, $\int \eta = 1$.

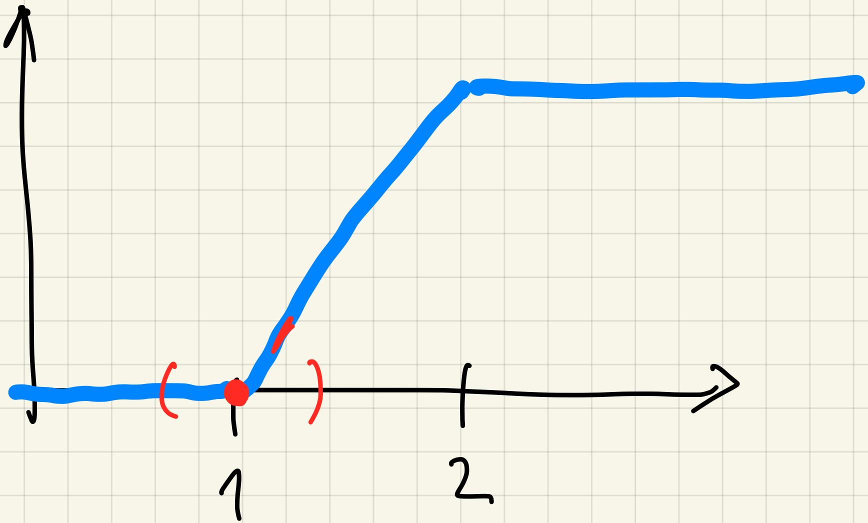
$\eta_\varepsilon(x) = \frac{1}{\varepsilon} \eta\left(\frac{x}{\varepsilon}\right)$

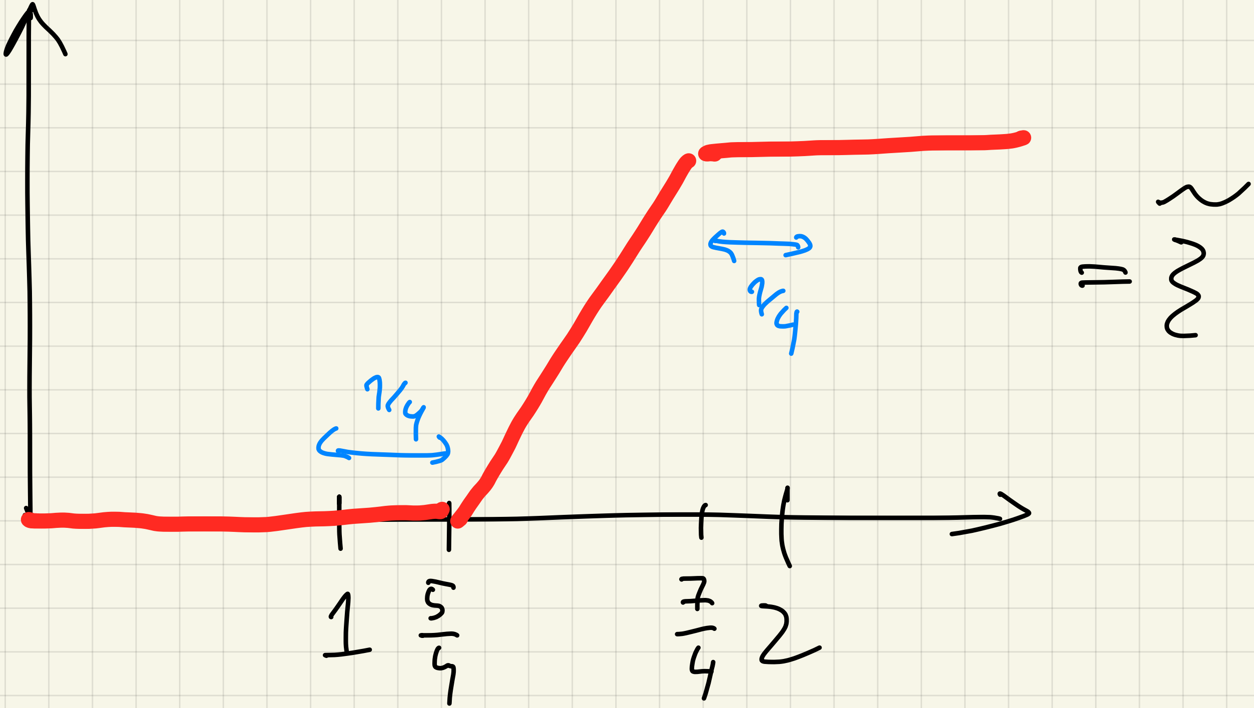
$\left(f \in L^p \qquad f * \eta_\varepsilon \in C^\infty \qquad f * \eta_\varepsilon \to f \text{ in } L^p. \right)$

- $\xi(x) = 0 \qquad |x| \leq 1$
- $\xi(x) = 1 \qquad |x| \geq 2$
- $|\xi'(x)| \leq C \qquad \forall x \in \mathbb{R}^+$





(to nie zadziała)



$$\xi(x) = \tilde{\xi} * \eta_{1/8}(x) = \int_{\mathbb{R}} \tilde{\xi}(y)\, \eta_{1/8}(x-y)\,dy .$$

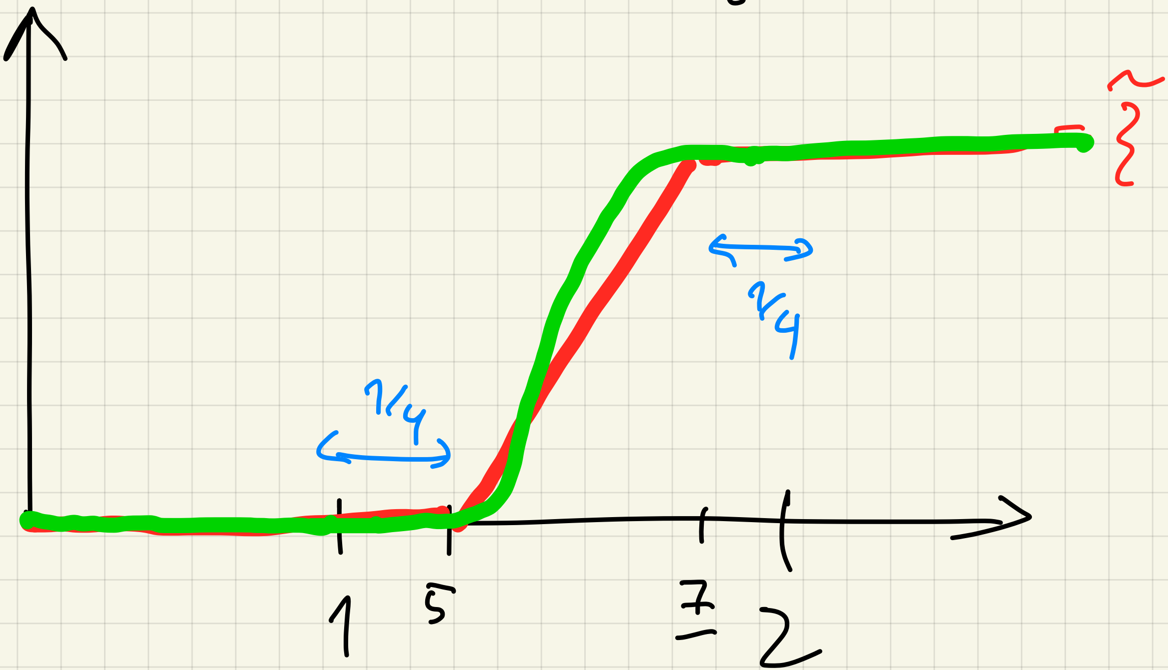
$$\xi(x) = \tilde{\xi} * \eta_{1/8}(x) = \int_{B_{1/8}(x)} \tilde{\xi}(y)\, \eta_{1/8}(x-y)\,dy .$$

Skoro η ma nośnik $B_1(0)$

to η_ε ma nośnik $B_\varepsilon(0)$

$$\xi(x) = \tilde{\xi} * \eta_{1/8}(x) = \int \tilde{\xi}(y)\, \eta_{1/8}(x-y)\, dy .$$

$$B_{1/8}(x)$$



TRANSPORTU

HARMON-1

CIEPŁA ;)

W ZWARTYM GRONIE ;).