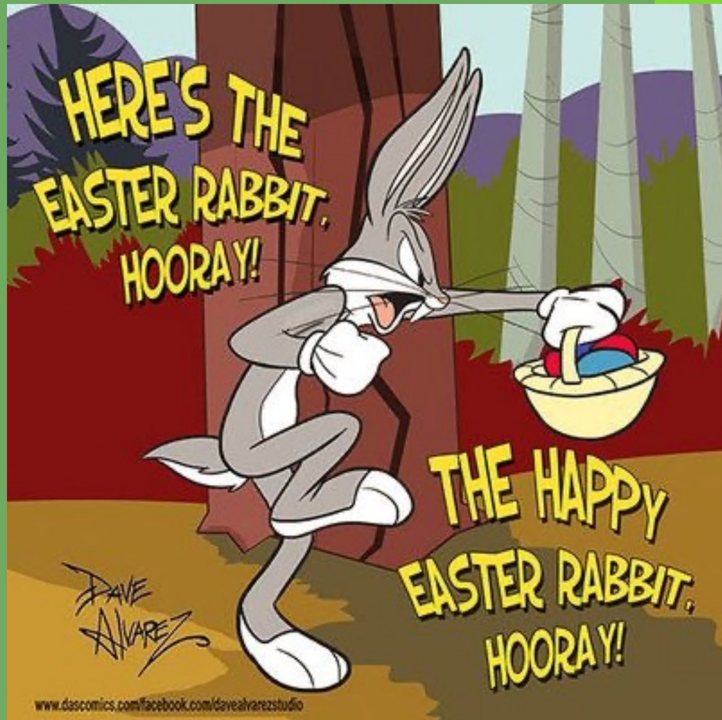
PDEs I: Tutorial 6.

15.04.2021



[Pytanie Patryka]

f jest jedn. ciągła to $\exists$ $\omega : \mathbb{R}^+ \to \mathbb{R}^+$

$\omega(0) = 0$

ω jest nierosnąca

ω jest ciągła w 0

$$|f(x) - f(y)| \le \omega(|x-y|)$$

$$\omega(t) = \sup_{|x-y| \le t} (f(x) - f(y))$$

$$\int_0^1 \frac{1}{\omega(t)}\, dt < \infty$$

(warunek Dini)

$$-\Delta u = f$$

$$u = \varphi$$

$$f \in C \Rightarrow u \in C^2$$

zad. 1 (B) (PS3)

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \qquad N(0, 2t)$$

↗ wariancje

$$\underbrace{u_t - \Delta u = 0}$$ na $\underbrace{\mathbb{R}^+}_{\text{czas } t} \times \underbrace{\mathbb{R}^n}_{\text{przestrzeń } x}$

$\to \quad N(0,2t) \Rightarrow \delta_0$

$$\forall_{\psi \in C_b(\mathbb{R})} \quad \int \psi(x)\, u(t,x)\, dx \longrightarrow \int \psi(x)\, d\delta_0 = \psi(0).$$

jak $t \to 0$.

$$\int \varphi(x)\, u(t,x)\,dx \longrightarrow \varphi(0). \qquad (t \to 0) \qquad \int u(t,x)\,dx = 1$$

D-d:

$$\left| \int \varphi(x)\, u(t,x)\,dx - \varphi(0) \right| = \left| \int \underbrace{[\varphi(x) - \varphi(0)]}_{\text{dla małych } x \text{ małe}} \underbrace{u(t,x)}_{\text{dla dużych } x \text{ małe}}\, dx \right|$$

bierzemy δ z ciągłości

$$\leq \int_{|x| \leq \delta} |\varphi(x) - \varphi(0)|\, u(t,x)\,dx + \int_{|x| > \delta} |\varphi(x) - \varphi(0)|\, u(t,x)\,dx$$

φ jest ciągłe więc np. na kuli jednostk. jest jedn. ciągłe.

Ust.

Ust. $\varepsilon > 0$. Istnieje $\delta > 0$ że $|\varphi(x) - \varphi(y)| \leq \varepsilon$

o ile $|x-y| \leq \delta$.

$$\int_{|x| \leq \delta} |\varphi(x) - \varphi(0)| \, u(t,x) \, dx \leq \varepsilon \int_{|x| \leq \delta} u(t,x) \leq \varepsilon.$$

$$\int_{|x| > \delta} |\varphi(x) - \varphi(0)| \, u(t,x) \, dx \leq 2\|\varphi\|_\infty \int_{|x| > \delta} u(t,x) \, dx$$

duże

$$= 2\|\varphi\|_\infty \int_{|x|>\delta} \underbrace{\frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}}_{t \to 0} \, dx$$

duże poza $x=0$.

małe

$$2\|\varphi\|_\infty \int_{\underset{r}{|x|>\delta}} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\, dx \overset{\uparrow}{=}$$

$$x = 2\sqrt{t}\, y$$
$$dx = (2\sqrt{t})^n dy$$

$$2\|\varphi\|_\infty \int_{|y|>\frac{\delta}{2\sqrt{t}}} \underbrace{\frac{1}{(4\pi t)^{n/2}}} e^{-|y|^2} \underbrace{(2\sqrt{t})^n}\, dy$$

$$= \underbrace{\frac{2\|\varphi\|_\infty\, 2^n}{(4\pi)^{n/2}}}_{\text{nie zal od } t} \underbrace{\int_{|y|>\frac{\delta}{2\sqrt{t}}} \overbrace{e^{-|y|^2}}^{\text{całk}}\, dy}_{t\to 0} \longrightarrow 0.$$

$$\left(-\infty, \frac{\delta}{2\sqrt{t}}\right) \cup \left(\frac{\delta}{2\sqrt{t}}, \infty\right)$$

$$\int_{|y| > \frac{\delta}{2\sqrt{t}}} e^{-|y|^2}\,dy = \int_{\mathbb{R}^n} \underline{\mathbb{1}_{|y| > \frac{\delta}{2\sqrt{t}}}\, e^{-|y|^2}}\,dy$$

$\longrightarrow 0$ punktowo jeśli $t \to 0$.

$\leq e^{-|y|^2}$ całkowalne.

Zasada maksimum: → CEL

Zad. 2 (PS 3)

$\Omega \subset \mathbb{R}^n \qquad u : \Omega \to \mathbb{R}$

(A) u osiąga maximum (minimum) w x^* we wnętrzu Ω to $Du(x^*) = 0$.

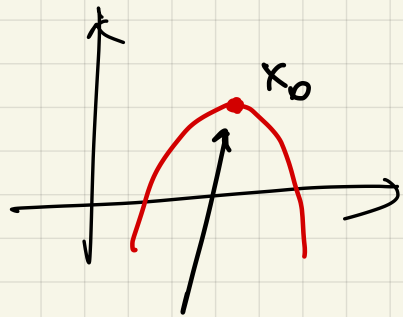
(jest w notatkach)

$D^2 u =$ macierz drugich poch.

(hesjan)

$$D^2 u(x^*) \leqslant 0 \quad \text{w sensie macierzy}$$

$$\forall_e \quad e^T D^2 u(x^*)\, e \leq 0.$$



wklęsła

$$f''(x_0) \leq 0.$$

Cel: x^* maksimum $\Rightarrow \forall_{e} \; e^T D^2 u(x^*) e \leq 0.$

$$f(x) = f(x^*) + \underbrace{Df(x^*)}_{=0}(x - x^*) + \frac{1}{2}(x-x^*)^T D^2 f(x^*)(x-x^*)$$

$$+ \; h(x)\,\|x - x^*\|^2$$

$$\left(h(x) \to 0 \quad x \to x^* \right)$$

$$\underbrace{\begin{pmatrix} \cdots\cdots\cdots \end{pmatrix}}_{= x^T} A \underbrace{\begin{pmatrix} \vdots \end{pmatrix}}_{\sim x}$$

$$f(x) = f(x^*) + \frac{1}{2}(x-x^*)^T D^2 f(x^*)(x-x^*) + h(x)\,\|x-x^*\|^2$$

x jest blisko $\cancel{x}^*$. $\qquad$ $\left(h(x) \to 0 \quad x \to x^*\right)$

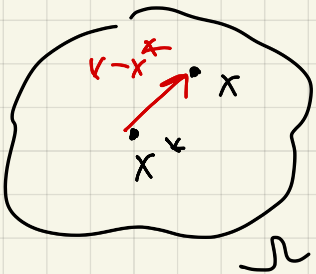
$$\underbrace{f(x) - f(x^*)}_{\leq 0} = \underbrace{\frac{1}{2}(x-x^*)' D^2 f(x^*)(x-x^*) + h(x)\,\|x-x^*\|^2}_{\leq 0}$$

$$f(x^*) \geq f(x)$$

$/ : \|x-x^*\|^2$

$$\frac{1}{2}\,\frac{x-x^*}{\|x-x^*\|}\,D^2 f(x^*)\,\frac{x-x^*}{\|x-x^*\|} + h(x) \leq 0.$$

$$\frac{1}{2}\,e^T D^2 f(x^*)\,e + h(x) \leq 0.$$



$$x \to x^* \Rightarrow \frac{1}{2}\,e^T D^2 f(x^*)\,e \leq 0.$$

$$\forall_{e} \quad e^T D^2 f(x^*)\, e \leq 0 \qquad \begin{pmatrix} \partial^2_{x_1 x_1} & \cdots & \partial^2_{x_1 x_n} \\ \vdots & & \\ \partial^2_{x_n x_1} & \cdots & \partial^2_{x_n x_n} \end{pmatrix}$$

$$\Downarrow$$

$$\partial^2_{x_i x_i} f(x^*) \leq 0 \quad \text{dla } e = e_i = (0, \dots, \overset{i}{1}, 0, \dots)$$

$$\Downarrow$$

$$\Delta f(x^*) \leq 0 ,$$

(B) $\Omega \subset \mathbb{R}^n$, $A \subset \Omega$ zwarty, f_n jest ciągłe

teza:

$f_n \rightrightarrows f$ na Ω $\Rightarrow$ Istnieje podciąg f_{n_k}

$$\sup_{x \in A} f_{n_k}(x) \to \sup_{x \in A} f(x).$$

D-d:

f_n jest ciągła $\sup_{x \in A} f_n(x) = f_n(x_n)$ $\exists_{x_n} \in A$.

$x_n \in A$, A jest zwarty $\Rightarrow \exists_{x_{n_k}}$ $x_{n_k} \to x^* \in A$.

Teza: $f(x^*) = \sup_{x \in A} f(x)$.

Wiem $x_{n_k} \longrightarrow x^*$

Chcę: $f_{n_k}(x_{n_k}) \longrightarrow f(x^*)$

$$|f_{n_k}(x_{n_k}) - f(x^*)| \leq |f_{n_k}(x_{n_k}) - f(x_{n_k})| +$$

$$+ |f(x_{n_k}) - f(x^*)| \leq \underbrace{\|f_{n_k} - f\|_\infty}_{\to 0 \quad f_{n_k} \rightrightarrows f} + \underbrace{|f(x_{n_k}) - f(x^*)|}_{\to 0}$$

bo f jest ciągła.

$$f(x^*) = \lim_{n_k \to \infty} f_{n_k}(x_{n_k}) \geq \lim_{n_k \to \infty} f_{n_k}(x) = f(x)$$

bo $f_{n_k} \rightrightarrows f$

więc $f_{n_k} \to f$ punktowo.

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u wybija max w x^* $\Rightarrow$ $\Delta u(x^*) \leq 0$.

$f_n \rightrightarrows f$ $\quad \sup_{x \in A} f_{n_k}(x) \longrightarrow \sup_{x \in A} f(x)$ $\quad \exists n_k$.

Problem 3

$$u_t - \Delta u \leq 0 \quad \text{na} \quad \underbrace{[0,T]}_{\text{czas}} \times \underbrace{\Omega}_{\text{przestrzeń}}$$

$\Rightarrow$ u wybija maksimum dla $t=0$ lub $x \in \partial\Omega$

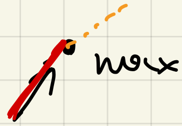
"parabolic boundary"

<u>D-d</u>: zacznijmy $u_t - \Delta u < 0$,

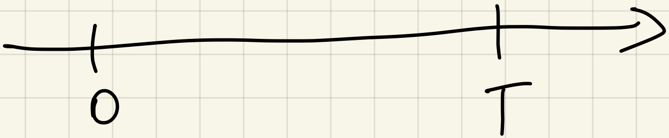
Załóżmy, że u wybija maksimum dla $t>0$, $x \in \Omega$

$\Delta u(t,x) \leq 0$

$t \in (0,T)$	$u_t = 0$
$t = T$	$u_t(t,x)$



$$u_t(T,x) \geq 0.$$



$$\Rightarrow \quad \Delta u(t,x) \leq 0$$

$$u_t \geq 0$$

$$\Rightarrow \quad u_t - \Delta u \geq 0$$

Sprzeczność bo

$$u_t - \Delta u < 0.$$

Ogólny przypadek:

$u_t - \Delta u \leq 0$

Rozważmy $v^\varepsilon(t,x) = u(t,x) - \varepsilon t$

$(+\varepsilon|x|^2)$

$v^\varepsilon_t = u_t - \varepsilon$

$\Delta v^\varepsilon = \Delta u$

$v^\varepsilon_t - \Delta v^\varepsilon \leq -\varepsilon < 0.$

$\Rightarrow$ dostajemy tezę dla v^ε (v^ε przyjmuje max na $t=0$ lub $x \in \partial\Omega$).

$v^\varepsilon \to u \qquad |v^\varepsilon - u| \leq \varepsilon t \leq \varepsilon T \to 0$

$$\sup_{(t,x)\in A} v^{\varepsilon}(t,x) \longrightarrow \sup_{(t,x)\in A} u(t,x) \qquad \forall_{A \text{ zwartego}}$$

$$A = \{ (t,x) : t = 0, \ x \in \partial\Omega)$$

$$A = \{ (t,x) : t \in [0,T], \ x \in \Omega \},$$