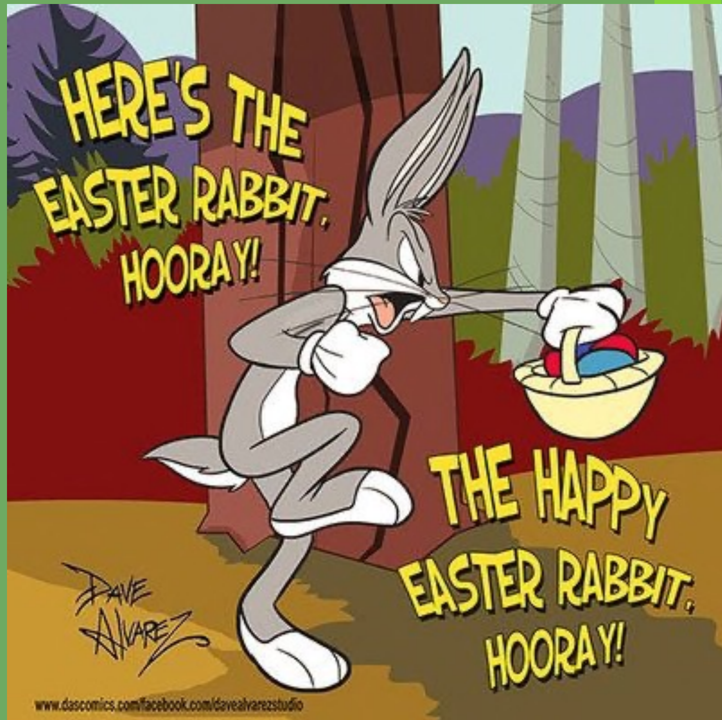
PDEs I: Tutorial 7.

22.04.2021



$\Uparrow$: $\quad u_t - \Delta u \underset{\geq}{\leq} 0 \quad$ in $\quad [0,T] \times \Omega$

$\Rightarrow$ <u>u przyjm. max. dla $t = 0$</u> lub $x \in \partial\Omega$.

$\Rightarrow$ <u>Zad. 4</u>

$$\begin{cases} u_t - \Delta u = f & (0,T] \times \Omega \\ u(0,x) = u_0(x) & t = 0 \\ u(t,x) = g(x) & x \in \partial\Omega \end{cases} \qquad (*)$$

<u>jednoznaczność</u> : Gdyby u, v spełniało $(*)$ to $\tilde{u} = u - v$

$$\left.\begin{array}{ll} \boxed{\tilde{u}_t - \Delta\tilde{u} = 0} & \\ \tilde{u}(0,x) = 0 & \tilde{u}(t,x) = 0 \\ t=0 & x \in \partial\Omega \end{array}\right| \Rightarrow$$

$$\begin{cases} \tilde{u} \text{ przyjmuje max które jest } 0. \\ \tilde{u} \text{ przyjmuje min które jest } 0. \end{cases}$$

$$\Rightarrow \quad \underline{\tilde{u} = 0.}$$

25 Zasada porównawcza

$$u_t - \Delta u = f_1 \qquad [0,T]\times\Omega$$

$$u(0,x) = u_0(x) \qquad t=0$$

$$u(t,x) = g_1(x) \qquad x\in\partial\Omega$$

$$v_t - \Delta v = f_2 \qquad [0,T]\times\Omega$$

$$v(0,x) = v_0(x) \qquad t=0$$

$$v(t,x) = g_2(x) \qquad x\in\partial\Omega$$

zał.

$f_1 \leq f_2$

$u_0 \leq v_0$

$g_1 \leq g_2$

$\Rightarrow u \leq v$

$$\tilde{u} = u - v$$

$$\tilde{u}_t - \Delta \tilde{u} = f_1 - f_2 \leq 0$$

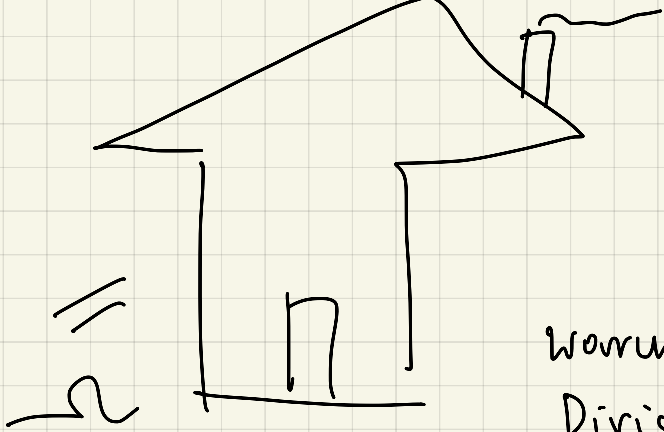
$$\tilde{u}(0,x) = u_0(x) - v_0(x) \leq 0$$

dla $x \in \partial\Omega$

$$\tilde{u}(t,x) = g_1(x) - g_2(x) \leq 0$$

$\tilde{u}$ przyjm. nak. dla $t=0$ lub $x \in \partial\Omega$.

$$\Rightarrow \tilde{u} \leq 0 \quad \Rightarrow \quad u \leq v$$



$$u_t - \Delta u = 0 \quad \Omega$$

$u(0,x) = u_0(x) \rightarrow$ wor. pocz.

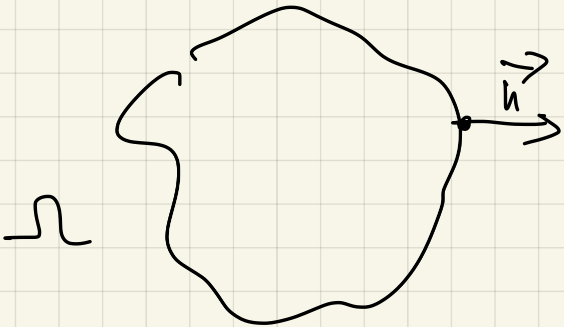
warunek Dirichleta

$$\boxed{u(t,x) = g(x) \quad \forall_{x\in\partial\Omega}}$$

(nierealistyczne)

Wor. brzegowy Neumanna:

$$\boxed{\frac{\partial u(x)}{\partial n} = g(x) \quad x \in \partial\Omega}$$



Zad. 7

$$(*)\quad \begin{cases} u_t - \Delta u = f & [0,T]\times\Omega \\ u(0,x) = u_0(x) & \text{war. pocz.} \\ \dfrac{\partial u(x)}{\partial n} = g(x) & x \in \partial\Omega \quad \text{war. brzeg.} \end{cases}$$

$$\frac{d}{dx} x^2 = 2x$$

Pok. że $(*)$ ma jedn. rozw.

$f=0$, $g=0$ w $(*)$, $E(t) = \int_\Omega |u(t,x)|^2 dx$

$$\frac{d}{dt}\int_\Omega |u(t,x)|^2 = \int_\Omega 2u\, u_t = \int_\Omega 2u\, \Delta u = -\int_\Omega |\nabla u|^2 \le 0$$

$f=0$

dla dow. $u, v \in C^2$

$$\int_\Omega \Delta u \cdot v + \int_\Omega \nabla u \cdot \nabla v = \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v \, dS(y)}_{\text{znika bo } \frac{\partial u}{\partial n} = 0}.$$

Puenta: $E(t)$ jest nierosnące.

<u>Jednoznaczność</u>: $\tilde{u} = u - v$ różnica rozwiązań.

$$E(t) \leq E(0) = 0 = \int_\Omega |u_0(x) - u_0(x)|^2.$$

$$w_f(x) = \int_{\Omega} \Phi(x-y)\, f(y)\, dy$$

vozw. "fundamentalne"

$$f \in C^{\alpha} \cap L^{\infty}(\Omega)$$

$$D_{i,j} w_f(x) = \int_{\Omega_0} D_{i,j} \Phi(x-y) \left(f(x) - f(y) \right) dy - f(x) \int_{\partial\Omega_0} D_i \Phi(x-y)\, \nu_j(y)\, dS(y)$$

j-ta wsp. wektora norm.

Chcemy: $-\Delta w_f = f \quad \Omega$

E6 $\quad \Delta w_f = ?$

$$D_{i,j}\, w_f(x) = \int_{\Omega_0} D_{i,j}\, \Phi(x-y) \left(f(x) - f(y) \right) dy - f(x) \int_{\partial\Omega_0} D_i \Phi(x-y)\, n_j(y)\, dS(y)$$

$$\Delta w_f(x) = \sum_{i=1}^{n} D_{i,i}\, w_f(x) = \sum_{i=1}^{n} \int_{\Omega_0} D_{i,i}\, \Phi(x-y) \left(f(x) - f(y) \right)$$

$$- f(x) \sum_{i=1}^{n} \int_{\partial\Omega_0} D_i \Phi(x-y)\, n_i(y)\, dS(y)$$

$$\sum_{i=1}^{n} \int_{\Omega_0} D_{i,i}\, \Phi(x-y) \big(f(x) - f(y) \big) =$$

$$= \int_{\Omega_0} \overline{\underline{\Delta \Phi(x-y)}}\, \overline{\big(f(x) - f(y) \big)}\, dy = 0$$

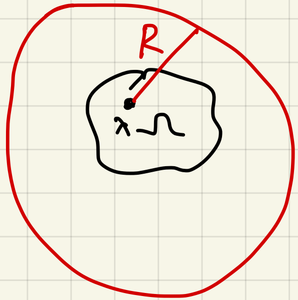
poza $x = y$.

$\Delta \Phi(z) = 0$

poza $z = 0$

$$-f(x)\sum_{i=1}^{n}\int_{\partial\Omega_0} D_i\Phi(x-y)\, n_i(y)\, dS(y)$$

$\Omega \subset \Omega_0$



$$= -f(x)\sum_{i=1}^{n}\int_{\partial B_R(x)} \underbrace{D_i\Phi(x-y)}_{=\ \frac{x_i-y_i}{n\alpha_n \underbrace{|x-y|^n}_{=R^n}}}\, n_i(y)\, dS(y)$$

$$= -f(x)\frac{1}{n\alpha_n R^n}\sum_{i=1}^{n}\int_{\partial B_R(0)} -y_i\, n_i\, dS(y).$$

$$= -f(x)\frac{1}{n\alpha_n R^n}\sum_{i=1}^{n}$$

$$= -f(x)\frac{1}{n\alpha_n R^n}\sum_{i=1}^{n}\int_{\partial B_R(0)} -y_i n_i \, dS(y).$$

$$= - f(x)\frac{1}{n\alpha_n R^n}\int_{\partial B_R(0)} -\sum_{i=1}^{n} y_i n_i \, dS(y)$$

$$= + f(x)\frac{1}{n\alpha_n R^n}\int_{\partial B_R(0)} \langle y, n\rangle \, dS(y)$$

$f(x).$

$\parallel$

$(y_1, y_2, \dots, y_n)$

$$= f(x)\frac{1}{n\alpha_n R^n}\int_{B_R(0)} \operatorname{div} y \, dy = f(x)\frac{1}{\alpha_n R^n}|B_R(0)|$$

$$\int_{\Omega}\operatorname{div} F\,dx = \int_{\partial\Omega}\langle F, n\rangle\, dS(y) \qquad \operatorname{div} y = \sum_{i=1}^{n}\partial_{y_i} y_i = n$$

E8

$f \in L^\infty \cap C^\alpha(\Omega)$ (nie obwiście jak $f \in C(\bar{\Omega})$)

$g \in C(\partial\Omega)$

$\Omega = B_R(0)$

istnieje dokł. jedno rozw. równania C^2

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega. \end{cases}$$

potrafimy znaleźć v_f t. że $-\Delta v_f = f$ (+)

potrafimy znaleźć $-\Delta v_g = 0$ Ω, $v_g = g$ $\partial\Omega$ dla Ω - kula.

potrafimy znaleźć w_f

t. że $-\Delta w_f = f$ $\oplus$

potrafimy znaleźć

$$-\Delta v_g = 0 \quad \Omega$$

$$\boxed{v_g = g} \quad \partial\Omega$$

dla Ω - kula.

$$\begin{cases} -\Delta w_f = f \\ \text{mam } w_f \end{cases} \qquad \begin{cases} -\Delta v = 0 & \Omega \\ v = \underbrace{g - w_f}_{\text{ciągłe}} & \partial\Omega \end{cases}$$

C^2

$\underline{u := w_f + v}$

$$\begin{cases} -\Delta u = f \\ u = w_f + g - w_f = g. \end{cases}$$

$\Rightarrow$ $\boxed{\text{Istnienie}}$

Jednoznaczność: z zasady maksimum.

(2-ga praca domowa),

E7 (odn. do wykł.)

$\Omega = \mathbb{R}^n$ $\quad f \in C^2$, f ma zwarty nośnik.

$$-\Delta u = f \quad \text{na } \mathbb{R}^n$$

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \Phi * f$$

$$= f * \Phi = \int_{\mathbb{R}^n} f(x-y)\, \Phi(y)\, dy.$$

f jest C^2.

$$\int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \underbrace{\int_{\operatorname{supp} f} \Phi(x-y)\, f(y)\, dy}$$

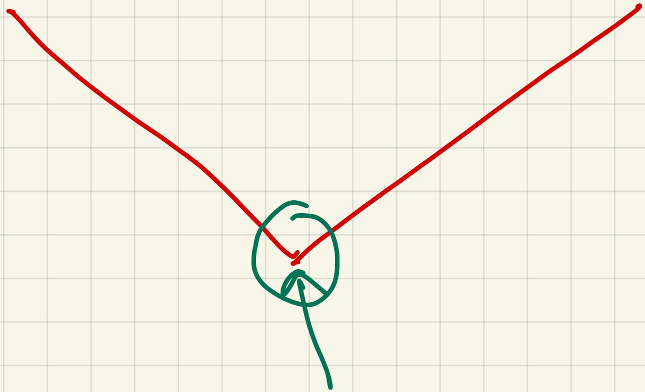
zwarty (ogr).

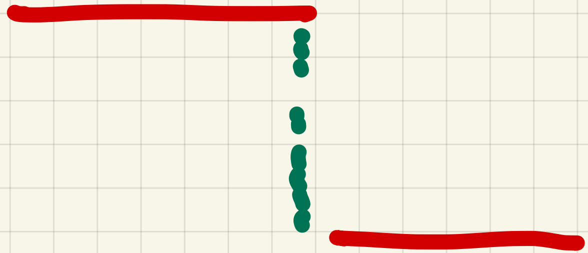
$f \in C^{\alpha}(\operatorname{supp} f)$

to ma regularność klasy C^2.

Dystrybucja:

Cel:





$\Omega \subset \mathbb{R}^n$

$V \subset \Omega$ zwarte to $C^\ell(V)$ jest p. Banacha z normą

$$\|f\|_{C^\ell(V)} = \sum_{k=0}^{\ell} \|D^k f\|_{L^\infty(V)}$$

! T liniowy funkcjonał na $C_c^\infty(\Omega)$

T jest dystrybucją jeżeli $\forall_{\substack{V \subset \Omega \\ V \text{ zwarty}}}$ istnieją C, ℓ

takie, że

$$|T(\varphi)| \leq C \, \|\varphi\|_{C^\ell(V)} \quad , \ \operatorname{supp} \varphi \subset V.$$

mogą zależeć od V

Stopień dystrybucji to najmniejsze l dla którego to oszacowanie zachodzi.

$= u \in L^1(K)$ dla każdego zwartego $K \subset \Omega$.

A1 $u \in L^1_{loc}(\Omega)$ zadaje dystrybucję

$$T_u(\varphi) = \int_\Omega \varphi(x)\, u(x)\, dx.$$

Ustalmy $V \subset \Omega$, V zwarty. Weźmy $\varphi \in C_0^\infty(\Omega)$, $\operatorname{supp} \varphi \subset V$.

$$|T_u(\varphi)| = \int_V \varphi(x)\, u(x)\, dx \leq \|\varphi\|_{C^0(V)} \|u\|_{L^1(V)}$$

stopień dystr. 0.

A2 $\mu \in \mathcal{M}^+(\Omega)$ $\quad |\mu(\Omega)| < \infty$

$$T_\mu(\varphi) = \int_\Omega \varphi(x)\, d\mu(x).$$

Ustalmy $V \subset \Omega$, V zwarty. Weźmy $\varphi \in C_0^\infty(\Omega)$, $\operatorname{supp} \varphi \subset V$.

$$|T_\mu(\varphi)| = \left| \int_V \varphi(x)\, d\mu(x) \right| \leq \|\varphi\|_{C^0(V)} \underbrace{\mu(V)}_{\leq \mu(\Omega)}.$$

Definicja (Schwarz)

Jeżeli T jest dystrybucją, α jest multiwskaźnikiem

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

$$D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$$

$D^\alpha T$ jest dystrybucją taką, że

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$$

B1 $(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$

$D^\alpha T$ jest dystrybucją.

Ust. $V \subset \Omega$, $\varphi \in C_0^\infty(\Omega)$ $\operatorname{supp} \varphi \subset V$.

$$|D^\alpha T(\varphi)| \leq |T(D^\alpha \varphi)| \leq C \|\varphi\|_{C^{\ell+|\alpha|}(V)}.$$

T jest dyst., więc istnieje C, ℓ

$$|T(\psi)| \leq C \|\psi\|_{C^\ell(V)}$$

B2 $\quad u \in C_c^\infty(\Omega)$, $\Omega \subset \mathbb{R}$

$\Omega =$ predintervalu $= (a,b)$

$$T_u(\varphi) = \int_{(a,b)} u(x)\,\varphi(x)\,dx$$

$$(\partial_x T_u)(\varphi) = (-1)\, T_u(\partial_x \varphi) =$$

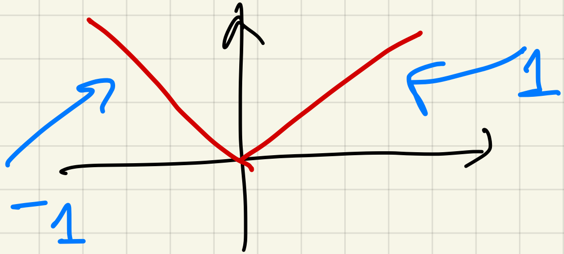
$$= (-1) \int_{[a,b]} u\, \partial_x \varphi \, dx = \int \partial_x u \cdot \varphi$$

$$= T_{\partial_x u}(\varphi).$$

T_u je

$$\int_a^b f' g + \int_a^b f g' = f g \Big|_a^b$$

(B3) $u(x) = |x|$. pochodna na $(-1, 1)$.



$$T_u(\varphi) = \int_{-1}^{1} \varphi(x)\, u(x) = \int_{-1}^{1} \varphi(x)\, |x|\, dx$$

$$= \int_{-1}^{0} \varphi(x)(-x)\, dx + \int_{0}^{1} \varphi(x)\cdot x\, dx$$

$$(\partial_x T_u)(\varphi) \underset{\text{def.}}{=} (-1)\, T_u(\partial_x \varphi) = -\int_{-1}^{0} \partial_x \varphi\, (-x)\, dx - \int_{0}^{1} \partial_x \varphi(x)\cdot x$$

$$-\int_{-1}^{0} \partial_x \varphi\,(-x)\,dx - \int_{0}^{1} \partial_x \varphi(x)\cdot x =$$

$$= -\ \varphi\cdot(-x)\Big|_{-1}^{0} + \int_{-1}^{0} \varphi(x)\,(-1)$$

$$-\ \varphi\cdot x\Big|_{0}^{1} + \int_{0}^{1} \varphi(x)\cdot 1 \;=\; \int_{-1}^{1} \varphi(x)\cdot \operatorname{sgn} x\,dx$$

$$\| $$

$$T_{\operatorname{sgn}(x)}(\varphi).$$

$$\int_a^b f'g + \int_a^b f g' = fg\Big|_a^b$$

$W \Uparrow$ policzacie $\Downarrow_{[0,\infty)}$.