Problem Set A1.

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Problem 1.

(A) $\frac{d}{dt} u(t, x+tb) = u_t(t, x+tb) + b\, u_x(t, -)$

$= 0 \Rightarrow t \mapsto u(t, x+tb)$ constant

$\Rightarrow \underset{\underset{u_0(x)}{\parallel}}{u(0,x)} = u(t, \underbrace{x+tb}_{y}) \Rightarrow x = y - tb$

$\Rightarrow u(t,y) = u_0(y-tb).$

(B) If u_1, u_2 are two C^1 solutions then $u_1 - u_2$ is a solution with initi. cond 0.

$\Rightarrow$ by (A) $u_1 - u_2 = 0$.

(C) We know $u^1(t,x) = u_0(x - tb_1)$
$u^2(t,x) = u_0(x - tb_2)$

$$|u^1 - u^2| \leq |u_0(x-tb_1) - u_0(x-tb_2)| \leq$$

$$\leq |u_0|_{Lip}\, t\, |b_1 - b_2|.$$

(D) Directly $|u^1(t,x) - u^2(t,x)| \leq \|u_0^1 - u_0^2\|_\infty.$

(E) $\|u\|_\infty \leq \|u_0\|_\infty$ from repr. formula

(F)

$$S_t f(x) = f(x - tb)$$

$$S_s S_t f(x) = S_s f(x - tb) = f(x - (t+s)b)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad S_{s+t} f(x).$$

Problem 2.

Exactly the same as Problem 1.

Problem 3.

Now, $\frac{d}{dt} u(t, x+tb) = cu(t, x+tb)$

$$\Rightarrow \frac{d}{dt}\left[u(t, x+tb)\, e^{-ct}\right] = 0$$

$$\Rightarrow u_0(x) = u(t, x+tb)\, e^{-ct}$$

$$\Rightarrow u(t, y) = u_0(y - tb)\, e^{ct}. \Rightarrow \text{existence}$$

For uniqueness: difference $\tilde{u} = u_1 - u_2$ solves

$$\partial_t \tilde{u} + \partial_x \tilde{u} \cdot b = c\tilde{u} \quad \text{with } \tilde{u}|_{t=0} = 0. \text{ By (A)}$$

we deduce $\tilde{u} = 0$.

(C) wrt b: (as before)

$$|u^1(t,x) - u^2(t,x)| \leq e^{ct} |b^1 - b^2|.$$

wrt c:

$$|u^1(t,x) - u^2(t,x)| = |u_0(x - tb)(e^{c_1 t} - e^{c_2 t})| \leq$$

if $u_0 \in L^\infty$.

$$\leq \|u_0\| e^{\max(c_1, c_2)t} t |c_1 - c_2|.$$

We used here Lipschitz continuity of e^x $\leftarrow$ $|x$.

(D) As above

$$|u^1 - u^2| \leq e^{ct} \|u_0^{(1)} - u_0^{(2)}\|_\infty.$$

(E) $\|u\|_\infty \leq e^{ct} \|u_0\|_\infty$.

(F) $S_t f = e^{ct} f(x - tb)$

$$S_s S_t f = e^{c(t+s)} f(x - (t+s)b) = S_{s+t} f.$$

$\square$.

Problem 4.

$$\begin{cases} \partial_t X_b(t,x) = b(X_b(t,x)) \\ X_b(0,x) = x \end{cases}$$

(A) X_b is globally well-defined: indeed, if the maximal time of existence T^* was finite, $|X_b(t,x)| \to \infty$ as $t \mapsto T^*$. But

$$|b(X_b(t,x))| \leq |b(0)| + |b|_{Lip} |X_b(t,x)|$$

$\Rightarrow$

$$|X_b(t,x)| \leq x + \int_0^t |b(X_b(s,x))| \, ds \leq$$

$$\leq x + t|b(0)| + \int_0^t |X_b(s,x)| \, ds$$

By Gronwall $|X_b(t,x)|$ grows exponentially with time but it's still bounded...

(B) We need $X_b(t, X_b(s,x)) = X_b(t+s, x)$.
Let $y(t) = X_b(t, X_b(s,x))$, $z(t) = X_b(t+s,x)$.
Note that $y(0) = z(0)$. We prove that y, z solve the same ODE with RHS Lipschitz. Then uniqueness for ODEs implies $y(t) = z(t)$.

$$y'(t) = \partial_t X_b(t, X_b(s,x)) = b(X_b(t, X_b(s,x)))$$
$$= b(y(t))$$

$$z'(t) = \partial_t X_b(t+s,x) = b(X_b(t+s,x)) = b(z(t)).$$

$\square$.

(C) $x \mapsto X_b(t,x)$ is Lipschitz continuous.

$$X_b(t,x) = x + \int_0^t b(X_b(s,x))\,ds$$
$$X_b(t,y) = y + \int_0^t b(X_b(s,y))\,ds$$

$$|X_b(t,x) - X_b(t,y)| \leq |x-y| + \int_0^t |b(X_b(s,x)) - b(X_b(s,y))|\,ds$$

$$\leq |x-y| + |b|_{Lip} \int_0^t |X_b(s,x) - X_b(s,y)|\,ds$$

$$\Rightarrow |X_b(t,x) - X_b(t,y)| \leq |x-y|\, e^{t|b|_{Lip}}.$$

(D) If $b \in C^1$ then $x \mapsto X_b(t,x)$ is C^1.

(This is theorem from ODEs on differantiability of solutions wrt initial conditions).

(E) $x \longmapsto X_b(t,x)$ is invertible and the inverse is simply $x \longmapsto X_b(-t,x)$.

This follows from (B) if $s=-t$.

(F) This follows from (E) together with (C) and (D).

Problem 5 $\frac{d}{dt} u(t, X_b(t,x)) = u_t(t, X_b(t,x)) +$

$+ \nabla_x u(t, X_b(t,x)) \cdot b(t, X_b(t,x)) = 0$

It follows that $u(t, X_b(t,x)) = u(0,x) = u_0(x)$.

Let $y = X_b(t,x) \Rightarrow x = X_b(-t,y)$. Then

$$u(t,y) = u_0(X_b(-t,y)). \qquad \square.$$

$\Uparrow$

Problem 6

Problem 7 Now, $\frac{d}{dt} u(t, X_b(t,x)) = c(X_b(t,x))\, u(t, X_b(t,x))$

$$\frac{d}{dt}\left[u(t, X_b(t,x))\, e^{-\int_0^t c(X_b(s,x))\,ds}\right] = 0$$

$$u_0(x) = u(t, \underbrace{X_b(t,x)}_{y})\, e^{-\int_0^t c(X_b(s,x))\,ds}$$

$$x = X_b(-t,y) \;, \quad X_b(s, X_b(-t,y)) = X_b(s-t,y).$$

$\square$

Problem 8. $\begin{cases} u_t + x u_x = 0 \\ u(0,x) = \cos x \end{cases}$ Initial condition

Suppose there is a curve $X(t,x)$ such that $t \mapsto u(t, X(t,x))$ is constant.

$$\frac{d}{dt} u(t, X(t,x)) = u_t(t, X(t,x)) + u_x(t, X(t,x))\, X_t(t,x)$$

From equation $u_t(t, X(t,x)) + u_x(t, X(t,x))\, X(t,x) = 0$

Try $X_t(t,x) = X(t,x)$ i.e. $X(t,x) = e^t \cdot x$

$$u(t, e^t x) = u(0,x) = \cos x \Rightarrow u(t,y) = \cos(y e^{-t}).$$

(we can check: $u_t + x u_x = -\sin(x e^{-t})(-e^{-t} x) +$
$+ x \cdot (-\sin(x e^{-t}))\, e^{-t} = 0$).

Problem 9

Problem 10

How to define generalized distributional solution? Start with

$$\begin{cases} \partial_t u(t,x) + \partial_x F(u(t,x)) = 0 \\ u(0,x) = u_0(x) \end{cases}.$$

Multiply with smooth $\varphi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R})$ and integrate:

- $$\iint \partial_t u(t,x)\, \varphi(t,x)\, dt\, dx =$$

$$= \int u(t,x)\varphi(t,x)\, dx \Big|_{t=0}^{t=\infty} - \iint u(t,x)\, \partial_t \varphi(t,x)\, dt\, dx$$

$$= -\int u_0(x)\, \varphi(0,x)\, dx - \iint u(t,x)\, \partial_t \varphi(t,x)\, dt\, dx.$$

- $$\iint \partial_x F(u(t,x))\, \varphi(t,x)\, dt\, dx =$$

$$= \iint F(u(t,x))\, \varphi(t,x)\, dt \Big|_{x=-\infty}^{x=+\infty} - \iint F(u(t,x))\, \partial_x \varphi(t,x)$$

$$= -\iint F(u(t,x))\, \partial_x \varphi(t,x).$$

so we obtain weak formulation. This is well-def. if u is only L^∞. All integrals are finite because $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$.

Now, suppose $u \in C^1_{t,x}$ and u is a distributional solution. Then, we can come back with integrals to obtain

$$\int [\partial_t u + \partial_x F(u)] \varphi(t,x) = 0 \quad \forall_{\varphi \in C_c^\infty}$$

$$\Rightarrow \partial_t u + \partial_x F(u) = 0$$

(We can also verify initial condition but for now it's quite technical...).

Problem 12

$$\partial_t \mu_t + b\, \partial_x \mu_t = 0$$

i.e. $\forall_{\varphi \in C_c}$

$$\int \partial_t \varphi \, d\mu_t \, dt + \int \partial_x (b\varphi) \, d\mu_t \, dt + \int \varphi(0,x) \, d\mu_0(x) = 0.$$

(A) If $d\mu_t = u(t,x)\,dx$ and u is C^1, we use integration by parts to deduce

$$\int \varphi(t,x)\left[\partial_t u(t,x) + b(x)\,\partial_x u(t,x)\right] = 0 \quad \forall \varphi .$$

(B) We propose $\mu_t = \delta_{x_0+bt}$.

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \partial_t \varphi(t,x)\,d\mu_t(x)\,dt = \int_{\mathbb{R}^+} \partial_t \varphi(t, x_0+bt)\,dt$$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \partial_x\big(b\,\varphi(t,x)\big)\,d\mu_t(x)\,dt = \int_{\mathbb{R}^+} b\,\partial_x\varphi(t, x_0+bt)\,dt$$

$+$

$$\int \frac{d}{dt}\varphi(t,x_0+bt)\,dt = \underset{\substack{\uparrow \\ \text{compact} \\ \text{support}}}{0} - \varphi(0,x_0) = -\int \varphi(0,x)\,d\mu_0(x)$$

$\square$.

(For $b(x)$ we have $\mu_t = \delta_{X_b(t,x)}$).