# Problem Set A2.

Kuba Skrzeczkowski

## Problem A1.

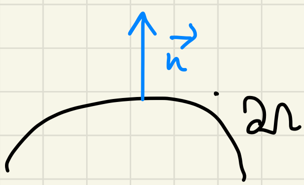
Divergence theorem : $F: \mathbb{R}^n \to \mathbb{R}^n$, $F \in C^1$,

$\to F(x) = (F_1(x), \dots, F_n(x))$.

$$\to \operatorname{div} F(x) := \partial_{x_1} F_1(x) + \dots + \partial_{x_n} F_n(x) =$$

$$= \nabla \cdot F = (\partial_{x_1}, \dots, \partial_{x_n}) \cdot F.$$

$$\to \int_\Omega \operatorname{div} F(x)\, dx = \int_{\partial\Omega} \underset{\substack{\| \\ F \cdot n}}{\langle F, n \rangle}\, ds$$




(A) $\Delta u = \operatorname{div}(\nabla u)$

$$\left( \sum \partial^2_{x_i} u = \sum \partial_{x_i} (\partial_{x_i} u) \right).$$

(B) Using divergence theorem

$$\int_\Omega \Delta u = \int_{\partial\Omega} \underset{\substack{\| \\ \nabla u \cdot n}}{\langle \nabla u, n \rangle} = \int_{\partial\Omega} \frac{\partial u}{\partial n}$$

## Problem A2

Using div thm with $F = v\nabla u$ we have

$$\operatorname{div}(v\nabla u) = \sum_{i=1}^{n} \partial_{x_i}(v\,\partial_{x_i} u) = \sum_{i=1}^{n} \partial_{x_i} v \cdot \partial_{x_i} u +$$

$$+ v \sum_{i=1}^{n} \partial_{x_i}^2 u = \nabla v \cdot \nabla u + v \cdot \Delta u. \qquad \square.$$

## Problem A3

Use Problem 2 twice.

## Problem A4

We consider $F = (0, 0, \dots, 0, \underset{j\text{-th}}{uv}, 0, \dots, 0)$.

Then, $\operatorname{div} F = D_j u \cdot v + u \cdot D_j v$. Moreover,

$\langle F, n \rangle = uv \cdot n_j$.

## Problem B1

$\Uparrow$

## Problem B2

(as martingales!!!)

Let $u$ be harmonic. Then

$$\Delta \varphi(u) = \sum_{i=1}^{n} \partial_{x_i}^2 \varphi(u) = \sum_{i=1}^{n} \partial_{x_i}\left[\varphi'(u)\partial_{x_i} u\right] =$$

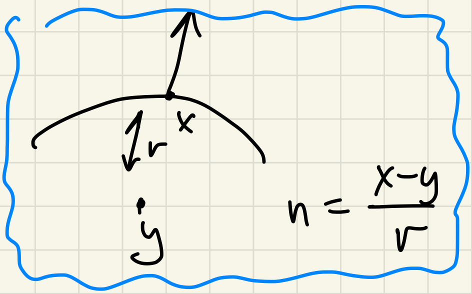
$$= \underbrace{\sum_{i=1}^{n} \varphi'(u)\partial_{x_i}^2 u}_{=0} + \underbrace{\varphi''(u)}_{\geq 0} \underbrace{|\nabla u|^2}_{\geq 0} \geq 0.$$

$\square$.

# Problem B3

$u$ is subharmonic i.e. $\Delta u \geq 0$. For all $B_R \subset \Omega$

$$0 \leq \int_{B_R(y)} \Delta u = \int_{\partial B_R(y)} \frac{\partial u}{\partial n} \, dS = \int_{\partial B_R(y)} \nabla u \cdot n \, dS(x) \Rightarrow$$




$$\int_{\partial B_r(y)} \nabla u(x) \cdot \frac{x-y}{r} \, dS(x) \geq 0.$$

Now, consider $\phi(r) := \fint_{\partial B(y,r)} u(x) \, dS$

$$\phi(r) = \frac{1}{\alpha(n) r^{n-1}} \int_{\partial B(y,r)} u(x) \, dS(x) =$$

$$\frac{x-y}{r} = z \qquad x = zr + y$$
$$dx = r^{n-1} dz$$

$$= \frac{1}{\alpha(n) r^{n-1}} \int_{\partial B(0,1)} u(zr+y) \, r^{n-1} \, dS(z)$$

$$= \fint_{\partial B(0,1)} u(zr+y) \, dS(z).$$

$$\phi(r) = \fint_{B(0,1)} u(zr+y)\, dS(z)$$

$$\phi'(r) = \fint_{B(0,1)} Du(zr+y)\cdot z\, dS(z) =$$

$$= \frac{1}{\alpha(n)} \int_{B(0,1)} Du(zr+y)\cdot z\, dS(z)$$

$x = zr+y$

$dx = r^{n-1} dz$

$z = \frac{x-y}{r}$

$$= \fint_{B(y,r)} \nabla u(x)\cdot \frac{x-y}{r}\, dS(z) \geq 0$$

so $\phi$ is non-decreasing in $r$. But $\phi(r) \to u(y)$ as $r \to 0$. So $\fint_{B(y,r)} u(x) dS \geq u(y)$.

To see 2nd formula $\int_{B(y,r)} u(x)dx =$

$$= \int_0^r \int_{\partial B(y,w)} u(x)\, dS(x)\, dw = \int_0^r \alpha(n) w^{n-1} \fint_{\partial B(y,w)} u(x) dS\, dw$$

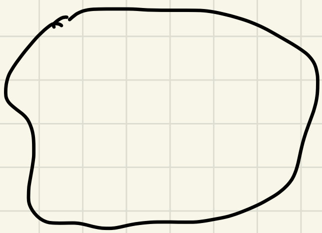
$$\geq u(y) \int_0^r \alpha(n) w^{n-1} dw = u(y)\, n\, \alpha(n)\, r^n.$$

$\square$.

For superharmonic function $u(y) \geq \fint_{\partial B_R(y)} u(x)\,dS(x)$.
For harmonic functions we have equality.

## Problem B4

$u$ is subharmonic in $\Omega$ and $u$ attains its max inside $\Omega$ in some $x_0 \in \operatorname{int} \Omega$. Then, for any $B_R(x_0)$



$$u\big|_{B_R(x_0)} = u(x_0) \text{ by B3.}$$

We cover $\Omega$ with such balls to conclude the proof.

## Problem B5

Clearly $u$ attains its max. somewhere in $\overline{\Omega}$. If this happens inside $\Rightarrow$ $u$ is constant in $\overline{\Omega}$, including the boundary. For harmonic function

$$\sup_{\Omega} u(x) = \sup_{\partial\Omega} u(x), \quad \inf_{\Omega} u(x) = \inf_{\partial\Omega} u(x).$$

## Problem B6

$\Delta u \geq 0$ so $u$ is subharmonic, i.e. $u$ attains sup on the boundary $\Rightarrow u \leq 0$.
If $u \geq 0$ then $u = 0$ (and this is a solution).

## Problem C1

If there were 2 solutions $u_1, u_2$ then

$$\begin{cases} \Delta(u_1-u_2)=0 & \Omega, \\ u_1-u_2=0 & \partial\Omega. \end{cases}$$

Then $u_1-u_2$ attains its max and min on $\partial\Omega$

So $u_1-u_2=0$. $\square$.

## Problem C2

Use A2 to $v=u=u_1-u_2$

$$\int (u_1-u_2)\,\underbrace{\Delta(u_1-u_2)}_{=0} + \int |\nabla (u_1-u_2)|^2 = \int_{\partial\Omega} \underbrace{(u_1-u_2)}_{=0 \text{ on } \partial\Omega} \frac{\partial (u_1-u_2)}{\partial n}$$

$$\Rightarrow \int_\Omega |\nabla(u_1-u_2)|^2 = 0 \Rightarrow u_1-u_2 \text{ is constant.}$$

Use value at $\partial\Omega$ to conclude $u_1=u_2$. $\square$.

## Problem C3

## Problem C4

If $u_1, u_2$ solve $\Delta u_1 = f$, $\Delta u_2 = f$ $\Rightarrow$

$\Delta(u_1 - u_2) = 0$ <u>in $\mathbb{R}^n$</u>

$\Rightarrow$ $u_1 - u_2$ is harmonic in $\mathbb{R}^n$

$\Rightarrow$ $u_1 - u_2 = C$ constant by Liouville Theorem

(so uniqueness is up to a constant $C$).

# Problem D1

$$\Phi(x) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x|^{2-n} & n > 2 \\ \frac{1}{2\pi} \log|x| & n = 2 \end{cases}$$

$$|x| = \sqrt{x_1^2 + \dots + x_n^2}$$

$$D_i |x| = \frac{1}{2} \frac{1}{\sqrt{x_1^2 + \dots + x_n^2}} 2x_i = \frac{x_i}{|x|}$$

$$D_i |x|^{2-n} = (2-n)|x|^{1-n} \frac{x_i}{|x|} = (2-n)|x|^{-n} x_i$$

$$D_i \log|x| = \frac{1}{|x|} \frac{x_i}{|x|} = \frac{x_i}{|x|^2}$$

$$D_i \Phi(x) = \begin{cases} \frac{1}{n\omega_n} \frac{x_i}{|x|^n} \\ \frac{1}{2\pi} \frac{1}{n\omega_n} \frac{x_i}{|x|^2} \end{cases} \leq \frac{1}{n\omega_n} |x|^{1-n}$$

$$D_{ij} \Phi(x) = \frac{1}{n\omega_n} \frac{\delta_{ij}|x|^n - x_i n |x|^{n-1} \frac{x_j}{|x|}}{|x|^{2n}} =$$

$$= \frac{1}{n\omega_n} \frac{\delta_{ij}|x|^n - n x_i x_j |x|^{n-2}}{|x|^{2n}} \leq \begin{cases} i \neq j & \frac{1}{\omega_n |x|^n} \\ i = j & \frac{1}{\omega_n |x|^n} \end{cases}.$$

## Problem D2

Fundamental solution in 1D:

$u'' = 0$ and $u$ is radially symmetric

$u$ is linear, $u = -\frac{1}{2}|x|$.

$$u'(x) = \begin{cases} -\frac{1}{2} & x > 0 \\ +\frac{1}{2} & x < 0 \end{cases}$$

"$u'' = -\delta_0$"

meaning: jump at $x = 0$.

Three meanings of $u''(x) = -\delta_0$

1) Function has jump of size 1 at one particular point.

2) $\int f(x)\, d u''(x) = -f(0)$

   so this is measure $-\delta_0$.

3) This is linear functional on space of cont. functions:

**Problem D3** "$\Delta \Phi(x) = -\delta_0(x)$".

i.e. $\int f(y)\, \Delta \Phi(y)\, dy = -\int f(x)\, \delta_0(x)\, dx$

$= -f(x).$

Green's 2nd identity:

$$\int_\Omega (v \Delta u - \Delta v\, u) = \int_{\partial\Omega} \left( v \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} u \right)$$

We apply this with $v(y) = \Phi(y-x)$, $x$ fixed.

$u(y) = u(y)$

Then $\int_\Omega \Delta \Phi(\underbrace{y-x}_{=0 \iff y=x}) \cdot u(y)\, dy = -\int_\Omega u(y)\, \delta_x\, dy$

$= -u(x).$

$$u(x) = -\int_\Omega \Phi(y-x)\, \Delta u(y)\, dy + \int_{\partial\Omega} \Phi(y-x) \frac{\partial u}{\partial n}\, dS(y)$$

$$- \int_\Omega \frac{\partial \Phi}{\partial n}(y-x) \cdot u(y)\, dS(y).$$

The rigorous proof in the lecture.

Problem D4 We know:

$$u(x) = -\int_{\Omega} \Phi(y-x)\,\Delta u(y)\,dy + \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial n}\,dS(y)$$
$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n}(y-x)\cdot u(y)\,dS(y).$$

troublemaker

Fix x and consider corrector solving for fixed x:

$$\begin{cases} \Delta \varphi^x = 0 & \Omega \\ \varphi^x = \Phi(x-y) & \partial\Omega \end{cases}$$

Then

$$\Phi(y-x)$$
$$\|$$
$$0 = -\int_{\Omega} \varphi^x(y)\,\Delta u(y) + \int_{\partial\Omega} \varphi^x(y)\frac{\partial u}{\partial n}\,dS(y)$$
$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n}(y-x)\,u(y)\,dS(y)$$

Hence

$$u(x) = -\int_{\Omega} \left[\Phi(y-x) - \varphi^x(y)\right] \Delta u(y)\, dy$$

$$- \int_{\partial\Omega} \frac{\partial}{\partial n} \left[\Phi(y-x) - \varphi^x(y)\right] u(y)\, dy.$$

This motivates GREEN FUNCTION for $\Omega$:

$$G(x,y) = \Phi(y-x) - \varphi^x(y).$$

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So far, we know that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$ then

$\Delta G(x,y) = 0$
$x \neq y.$

$$u(x) = \int_{\Omega} G(x,y) f(y)\, dy - \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n_y} g(y)\, dy.$$

BUT DOES IT EXISTS?

Now, we recall some facts from the lecture:

① $G(x,y) = G(y,x).$

② If $\Omega = B_r(0)$ then one can compute $G$ to obtain $G$ s.t. $\dfrac{\partial G(x,y)}{\partial n_y} = -\dfrac{r^2-|x|^2}{n\alpha_n r}\dfrac{1}{|x-y|^n}$.

In particular, when $\begin{cases} -\Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$ we know that

$$u(x) = \int_{\partial B(0,r)} \frac{r^2-|x|^2}{n\alpha_n r} \frac{g(y)}{|x-y|^n} \, dS(y).$$

## Problem D5

Using Green function one can obtain formula for solution to $\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$ $\Omega = B_r(0)$.

$$u(x) = \frac{r^2-|x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dy$$

$$= -\int \frac{\partial G(x,y)}{\partial n_y} g(y) \, dy$$

$$\frac{\partial G(x,y)}{\partial n_y} = -\frac{r^2-|x|^2}{n\alpha_n r} \frac{1}{|x-y|^n}.$$

We will prove:

(a) $u \in C^\infty(B_r(0))$

(b) $\Delta u = 0$ in $B_r(0)$

(c) $\forall_{x_0 \in \partial B_r(0)} \lim_{\substack{x \to x_0 \\ x \in B_r(0)}} u(x) = g(x_0)$.

<u>Proof of (a) and (b):</u>

For fixed $x$, the map $y \mapsto G(x,y)$ is harmonic except $y=x$. As $G(x,y) = G(y,x)$, the map $x \mapsto G(x,y)$ is harmonic, except $x=y$. Hence $x \mapsto \frac{\partial G}{\partial n_y}(x,y)$ is harmonic. Hence

$$\Delta_x \frac{\partial G}{\partial n_y}(x,y) = 0 \qquad x \in B_r(0), \; y \in \partial B_r(0)$$

Directly from the formula we see that

$$\left\| \Delta_x \frac{\partial G}{\partial n_y} \right\|_{L^\infty(B_{r(1-\varepsilon)}(0))} \text{ is bounded}$$

so DCT shows $\forall x \in B_{r(1-\varepsilon)}(0)$ $\Delta u(x) = 0$.
Finally, we send $\varepsilon \to 0$ to conclude the proof.

<u>Proof of (c):</u>
Note that $u(x) = 1$ is harmonic so applying theory above

$$1 = u(x) = \int_{\partial B_r} -\frac{\partial G}{\partial n_y}(x,y) \cdot 1 \, dy = -\int_{\partial B_r} \frac{\partial G}{\partial n_y}(x,y) dy$$

Hence, fix $\varepsilon > 0$, choose $\delta$ $|g(x_0) - g(y)| \leq \varepsilon$ for $|x_0 - y| \leq \delta$
If $|x_0 - x| \leq \delta/2$ then

$$g(x_0) - u(x) = -\int_{\partial B_r(0)} (g(x_0) - g(y)) \frac{\partial G}{\partial n_y}(x,y) dy$$

$$= -\int_{\partial B_r(0) \cap |x_0 - y| \leq \delta} (g(x_0) - g(y)) \frac{\partial G}{\partial n_y}(x,y) dy$$

$$- \int_{\partial B_r(0) \cap |x_0 - y| > \delta} (g(x_0) - g(y)) \frac{\partial G}{\partial n_y}(x,y) dy$$

The first integral is easily estimated with $\varepsilon$ because $\int_{\partial B_r} \frac{\partial G}{\partial n_y}(x,y)\,dy = 1$.

For the second we observe that

$$\delta < |x_0 - y| \leq |x_0 - x| + |x-y| \leq \frac{\delta}{2} + |x-y|$$

so $\delta/2 < |x-y|$. It follows that the second term is bounded with

$$\dots \leq \frac{C}{(\delta/2)^n}\left(r^2 - |x|^2\right) = \frac{C}{(\delta/2)^n}\left(r-|x|\right)\left(r+|x|\right) \leq$$

$$\leq \frac{C\,(2r)}{(\delta/2)^n}\left(r-|x|\right) \leq \frac{C\,(2r)}{(\delta/2)^n}|x-x_0| \leq \varepsilon$$

if $|x-x_0| \leq \frac{\varepsilon(\delta/2)^n}{C\,2r}$ so if

$$|x-x_0| \leq \min\left\{ \delta/2,\ \frac{\varepsilon(\delta/2)^n}{C2r} \right\}$$

the second term is bounded by $\varepsilon$. $\square$.

Problems D6, D7

## PART E: EXISTENCE FOR $\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$

Motivation: if $\Delta_x G(x-y) = -\delta_{y=x}$ then

$\int_\Omega f(y) G(x-y)$ should solve Poisson's equation because

$$\Delta_x \int_\Omega f(y) G(x-y) = \int_\Omega f(y) \Delta_x G(x-y)\, dy =$$

$$= -\int_\Omega f(y) \delta_{y=x}\, dy = -f(x)$$

Therefore, we expect that $\Delta \int_\Omega G(x-y) f(y)\, dy = -f(x)$.

Rigorous proof: requires careful analysis of singularity in $G(x-y)$ and some assumptions on $f$

As we will see, the minimal assumption will be $f \in C^\alpha$ $\alpha \in (0,1]$.

**Problem E1** There is smooth $\xi : \mathbb{R} \to \mathbb{R}$ s.t.

- $\xi(x) = 0$ for $|x| \leq 1$
- $\xi(x) = 1$ for $|x| \geq 2$
- $|\xi'(x)| \leq C$ $\forall_x$.

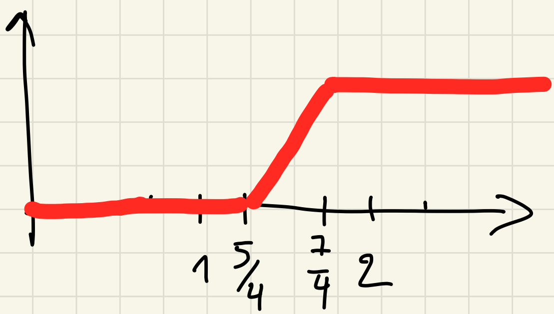
Recall mollifiers: We consider $\eta_\varepsilon$ to be standard mollifying kernel i.e.

$$\eta \text{ s.t. } \int \eta = 1, \quad \eta \geq 0, \quad \operatorname{supp} \eta \subset B(0,1).$$

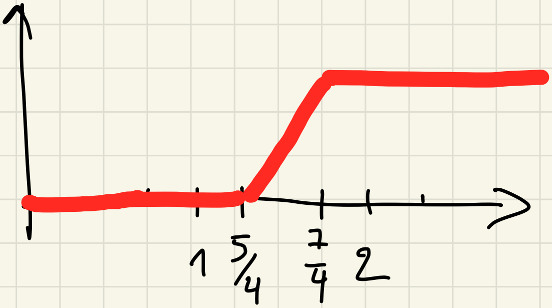
$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right)$$

Then ( $f * \eta_\varepsilon \longrightarrow f$ in $C[0,1]$, $L^p(0,1)$ $(1 \leq p < \infty)$ )

We consider

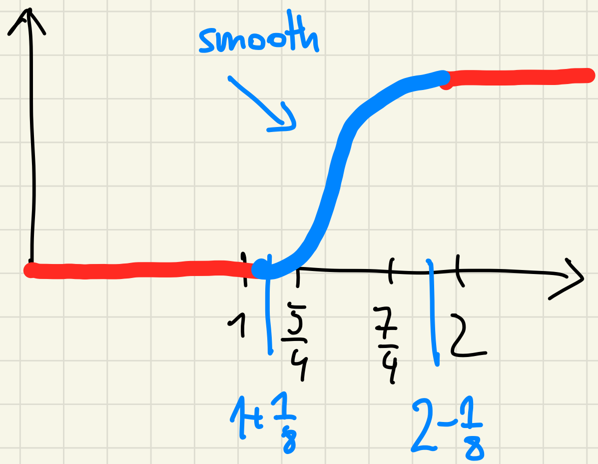
$\tilde{\xi} =$ 

$\tilde{\xi} =$




Then $\tilde{\xi} * \eta_{1/8}(x) = \int_{B(x,\frac{1}{8})} \tilde{\xi}(y)\, \eta_{1/8}(x-y)\, dy$

$\Rightarrow \tilde{\xi} * \eta_{1/8} =$




$|(\tilde{\xi} * \eta_{1/8})'| \leq C \to$ continuous function which vanishes outside $[1,2]$.

**Problem E2** $\quad \zeta_\varepsilon(x) = \zeta\left(\frac{|x|^2}{\varepsilon^2}\right)$

Then $\quad \zeta_\varepsilon(x) = 0$ when $|x| \leq \varepsilon$

$\qquad\quad \zeta_\varepsilon(x) = 1$ when $|x| \geq \sqrt{2}\,\varepsilon$

$$D\zeta_\varepsilon(x) = \zeta'\left(\frac{|x|^2}{\varepsilon^2}\right) \cdot \frac{1}{\varepsilon^2}\, 2x$$

$$= \underbrace{\frac{2x}{\varepsilon}}_{\leq \sqrt{2}} \cdot \zeta'\left(\frac{|x|^2}{\varepsilon^2}\right) \cdot \frac{1}{\varepsilon} \leq \frac{C}{\varepsilon}$$

**Problem E3**

Claim: $\quad w_f(x) := \int_\Omega \Phi(x-y) f(y)\, dy$

$$w_f^\varepsilon(x) := \int_\Omega \Phi(x-y)\, \zeta_\varepsilon(x-y)\, f(y)\, dy$$

$w_f^\varepsilon \to w_f$ uniformly in $\Omega$

Proof:

$$\left| w_f(x) - w_f^{\varepsilon}(x) \right| = \left| \int_{\Omega} \Phi(x-y) f(y) \underbrace{\left[ 1 - \zeta^{\varepsilon}(x-y) \right]}_{\substack{=0 \text{ when} \\ \zeta^{\varepsilon}(x-y)=1 \text{ when} \\ |x-y| \geq \sqrt{2}\,\varepsilon}} dy \right|$$

$$\leq \left| \int_{|x-y| \leq \sqrt{2}\varepsilon} \underbrace{\Phi(x-y)}_{\frac{C}{|x-y|^{n-2}}} f(y)\, dy \right| \leq \left| \int_{|x-y| \leq \sqrt{2}\varepsilon} \frac{C}{|x-y|^{n-2}} f(y)\, dy \right|$$

$$\leq \int_{B(x,\sqrt{2}\varepsilon)} \frac{C}{|x-y|^{n-2}} |f(y)|\, dy \leq C \|f\|_{\infty} \int_0^{\sqrt{2}\varepsilon} \frac{1}{r^{n-2}} r^{n-1}\, dr$$

$$\leq C \|f\|_{\infty} \varepsilon^2 \longrightarrow 0$$

Case $d=2$ $\Rightarrow$ SIMILAR $\left( \int_0^{\varepsilon} r \log r \, dr \right)$

Problem E4

Target: compute derivative of $\int_\Omega \Phi(x-y) f(y) dy$

Claim: $D^i w_f(x) = \int_\Omega D^i \Phi(x-y) f(y) dy$

Step 1: this function is well-defined, i.e. for all $x \in \Omega$ $D^i w_f(x) < \infty$. Proof: find $R_x$ s.t. $\Omega \subset B_{R_x}(x)$ so that

$$|D^i w_f(x)| \leq \int_{B_{R_x}(x)} |D^i \Phi(x-y)| \, \|f\|_\infty dy$$

$$\leq \|f\|_\infty \int_0^{R_x} \frac{C}{r^{n-1}} r^{n-1} dr \leq C \|f\|_\infty R_x < \infty$$

Step 2: $$D^i w_f^\varepsilon(x) = \int_\Omega D^i \left[\Phi(x-y) \zeta^\varepsilon(x-y)\right] f(y) dy$$

$$= \int_\Omega D^i \Phi(x-y) \underbrace{f(y)}_{\zeta^\varepsilon(x-y)} dy + \int_\Omega \Phi(x-y) D^i \zeta^\varepsilon(x-y) f(y) dy$$

how to differentiate integrals:

$$\frac{d}{dx}\int_\Omega F(x,y)\,dy \stackrel{?}{=} \int_\Omega \frac{d}{dx}F(x,y)\,dy \quad ? \quad (*)$$

$$\frac{1}{h}\left[\int_\Omega F(x+h,y)\,dy - \int_\Omega F(x,y)\,dy\right] =$$

$$= \int_\Omega \frac{F(x+h,y)-F(x,y)}{h}\,dy$$

$$\downarrow$$

$$\frac{d}{dx}F(x,y) \text{ pointwisely}$$

Note $\dfrac{F(x+h,y)-F(x,y)}{h} \leq \|F_x\|_\infty$ so if

we integrate over the set of finite measure and derivative is bounded we are allowed to conclude $(*)$.

## Second integral:

$$\left| \int_\Omega \Phi(x-y) D^i \zeta^\varepsilon(x-y) f(y)\,dy \right| \leq$$

$$\leq \frac{1}{\varepsilon} \int_{\varepsilon \leq |x-y| \leq \sqrt{2}\varepsilon} |\Phi(x-y)|\,|f(y)|\,dy \leq \|f\|_\infty \frac{1}{\varepsilon} \int_{\varepsilon \leq |x-y| \leq \sqrt{2}\varepsilon} \frac{C}{|x-y|^{n-2}}$$

$$\leq C \|f\|_\infty \frac{1}{\varepsilon} \int_\varepsilon^{\sqrt{2}\varepsilon} \frac{r^{n-1}}{r^{n-2}}\,dr \leq C \|f\|_\infty \varepsilon \to 0$$

uniformly in $x$

## First integral:

$$\left| \int_\Omega D^i \Phi(x-y) \left[ \zeta^\varepsilon(x-y) - 1 \right] f(y)\,dy \right| \leq$$

$$\leq \int_{|x-y| \leq \sqrt{2}\varepsilon} |D^i \Phi(x-y)|\,\|f\|_\infty\,dy \leq$$

$$\leq C \|f\|_\infty \int_{|x-y| \leq \sqrt{2}\varepsilon} \frac{1}{|x-y|^{n-1}}\,dy \leq C \|f\|_\infty \int_0^{\sqrt{2}\varepsilon} \frac{1}{r^{n-1}} r^{n-1}\,dr$$

$$= C \|f\|_\infty \sqrt{2}\varepsilon \to 0$$

again, uniformly in $x \in \Omega$.

Conclusion:

$$w_f^\varepsilon \rightrightarrows w_f \quad \text{in } \Omega$$

$$D_i w_f^\varepsilon \rightrightarrows \int_\Omega D_i \Phi(x-y) f(y) \quad \text{in } \Omega$$

$\Rightarrow$ for all $\tilde{\Omega}$ compact, $\tilde{\Omega} \subset \Omega$, $w_f^\varepsilon$ is Cauchy in $C^1(\tilde{\Omega})$ = Banach space.

$\Rightarrow w_f^\varepsilon \longrightarrow g$ in $C^1(\tilde{\Omega}) \Rightarrow g = w_f \Rightarrow$

$$w_f \in C^1(\tilde{\Omega})$$

$D_i w_f^\varepsilon \longrightarrow D_i w_f$ in $C(\tilde{\Omega}) \Rightarrow$

$$D_i w_f = \int_\Omega D_i \Phi(x-y) f(y)\, dy$$

# Problem E5

How to compute 2nd derivative?

$$D_{ij} w_f(x) = \int_\Omega D_{ij} \Phi(x-y) f(y)\, dy \quad ?$$

This is not well-defined for $f \in L^\infty$.

$$|D_{ij}\Phi(x-y)| \leq \frac{C}{|x-y|^n}$$

But $\int_\Omega \frac{C}{|x-y|^n} dy \leq \int_{B_R(x)} \frac{C}{|x-y|^n} dy \approx \int_0^R \frac{1}{r^n} r^{n-1} dr = \infty$

This motivated looking for conditions on $f$ allowing to compute $D_{ij} w_f$. The condition is Hölder continuity

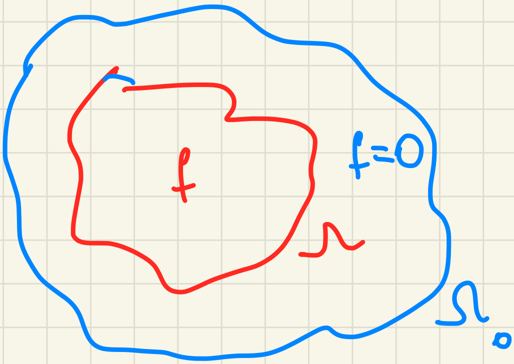
i.e. $\exists_C \quad |f(x) - f(y)| \leq C|x-y|^\alpha$.

$\Uparrow$:

E4. Let $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$ for some $\alpha \in (0,1]$. Consider $\Omega_0$ such that $\Omega \subset \Omega_0$ and extend $f = 0$ in $\Omega_0 \setminus \Omega$. Then $w_f \in C^2(\Omega)$ and

$$D_{i,j} w_f(x) = \int_{\Omega_0} D_{i,j}\Phi(x-y)\,(f(x) - f(y))\,\mathrm{d}y - f(x) \int_{\partial\Omega_0} D_i\Phi(x-y)\, n_j(y)\,\mathrm{d}S(y)$$

where $n_j$ is the $j$-th component of $\mathbf{n}$.




Now, this term is well-def because $|f(x) - f(y)| \leq C|x-y|^\alpha$.

Schauder estimates for $-\Delta$: $(-\Delta u = f)$

$f \in C^0 \;\cancel{\Longrightarrow}\; u \in C^2$

$f \in C^1 \Longrightarrow u \in C^{2+1}$ (Schauder)

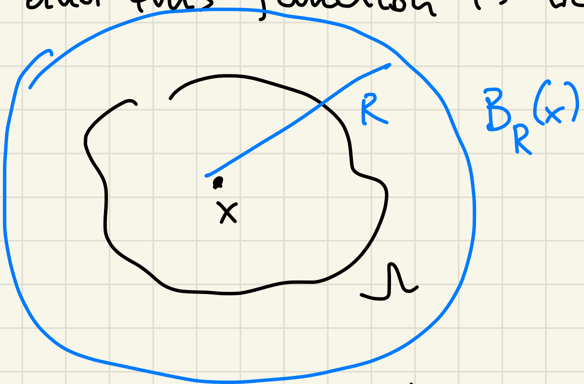
$f \in L^2 \Longrightarrow u \in H^2$ (future)

(0 weak derivatives) (two weak derivatives)

# Problem E6

$$D_{i,j} w_f(x) = \int_{\Omega_0} D_{i,j} \Phi(x-y)\big(f(x)-f(y)\big)\, dy - f(x) \int_{\partial\Omega_0} D_i \Phi(x-y)\, n_j(y)\, dS(y)$$

and this function is well-defined.




$\Omega \subset B_R(x) := \Omega_0$

$$D_i \Phi(x) = \frac{x_i}{n \alpha_n |x|^n}.$$

$$\sum_{i=1}^n D_{ii} w_f(x) = -\sum_{i=1}^n f(x) \int_{\partial B_R(x)} D_i \Phi(x-y)\, n_i(y)\, dS(y)$$

$$= -f(x) \sum_{i=1}^n \int_{\partial B_R(x)} \frac{x_i - y_i}{n \alpha_n |x-y|^n} n_i \, dS(y) =$$

$$= -f(x) \int_{\partial B_R(0)} \frac{\langle y, n \rangle}{n \alpha_n |y|^n} \, dS(y)$$

$$-f(x)\int_{\partial B_R(0)} \frac{\langle y, n\rangle}{n\alpha_n |y|^n}\, dS(y) =$$

$$= -f(x)\frac{1}{n\alpha_n R^n}\int_{\partial B_R(0)} \langle y, n\rangle\, dS(y) =$$

$$= -f(x)\frac{1}{n\alpha_n R^n}\int_{B_R(0)} \underbrace{\operatorname{div}(y)}_{n}\, dx = -f(x).$$

$\square$.

**Problem E7** In the lecture $\Omega = \mathbb{R}^n$ and $f \in C^2_c$ then $\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$ solves Poisson equation.

$$\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy$$

$f \in C^2_c$ so can be directly differentiated.

Note that if $f \in C^\alpha$ and $f$ is comp. supp. we can write

$$\int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy = \int_{\operatorname{supp} f} \Phi(x-y) f(y)\, dy$$

and we can use our theory to see that $u \in C^2$ (even $C^{2,\alpha}$ by Schauder estimate).

**Problem E8** Uniqueness to $\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$ is clear. For existence solve

$$\begin{cases} -\Delta v = 0 & \Omega \\ v = g - w_f & \partial\Omega \end{cases}$$ then $u = v + w_f$ solves .