Problem Set A3

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Problem 1

$x_1^2 + \dots + x_n^2$

(4) $$u(t,x) = \frac{1}{4(\cancel{2}\pi t)^{n/2}} e^{-|x|^2/\cancel{2}4t}$$

$$u_t = \frac{-n/2}{(4\pi t)^{n/2+1}} 4\pi e^{-|x|^2/4t} +$$

$$+ \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \left(+ \frac{|x|^2}{4t^2} \right)$$

$$= \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/2t} \left[\frac{-\frac{n}{2} 4\pi}{4\pi t} + \frac{|x|^2}{4t^2} \right]$$

$$= u(t,x) \left[-\frac{n}{2t} + \frac{|x|^2}{4t^2} \right]$$

$$\partial_{x_i} u = u(t,x) \cdot \left(\frac{-2x_i}{4t} \right) = u(t,x) \frac{(-x_i)}{2t}$$

$$\partial^2_{x_i} u = -\frac{1}{2t} u(t,x) + u(t,x) \cdot \left(\frac{-2x_i}{4t} \right) \left(\frac{-x_i}{2t} \right)$$

$$= -\frac{u(t,x)}{2t} + u(t,x) \frac{x_i^2}{4t^2}$$

$$\sum \partial^2_{x_i} u = u(t,x) \left[-\frac{n}{2t} + \frac{|x|^2}{4t^2} \right].$$

✓.

(B) $N(0,2t) \Longrightarrow \delta_0$ as $t \to 0$

$\forall f$ cont, bound $\quad \underset{\substack{N(0,t) \\ \|}}{\mathbb{E}} f \overset{?}{=} \mathbb{E}_{\delta_0} f = f(0)$

$$\int f(x)\, u(t,x)\, dx$$

As $\int u(t,x)\, dx = 1$ we write

$$\left| f(0) - \int f(x) u(t,x) dx \right| = \left| \int [f(0)-f(x)]\, u(t,x)\, dx \right|$$

$$\leq \int_{|x|\leq\delta} |f(0)-f(x)|\, u(t,x)\, dx + \int_{|x|\geq\delta} |f(0)-f(x)|\, u(t,x)\, dx$$

Fix $\varepsilon > 0$. Choose $\delta > 0$ s.t. $|x-0| \leq \delta \Rightarrow |f(x)-f(0)| \leq \varepsilon$.

Then, (FIRST INT) $\leq \varepsilon$. For the second we estimate

$$\int_{|x|\geq\delta} |f(0)-f(x)|\, u(t,x)\, dx \leq 2\|f\|_\infty \int_{|x|\geq\delta} u(t,x)\, dx$$

$$\leq 2\|f\|_\infty \int_{|x|\geq\delta} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\, dx$$

(Note: dominated convergence WILL NOT work!)

We change variables:

$$\int_{|x|\geq\delta} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\, dx \underset{\substack{x = 2\sqrt{t}\,y \\ dx = (4t)^{n/2}\,dy}}{\overset{\uparrow}{=}} \int_{|y|\geq \delta/2\sqrt{t}} \frac{1}{\pi^{n/2}} e^{-|y|^2} dy$$

$$= \int_{\mathbb{R}^n} \mathbb{1}_{|y|\geq \delta/2\sqrt{t}} \frac{1}{\pi^{n/2}} e^{-|y|^2/2}\, dy.$$

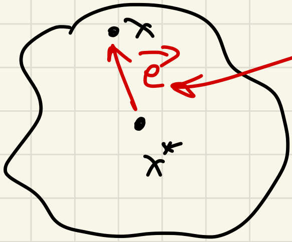
Now, we can use dominated convergence. $\square$

Problem 2

1) $f(x) = f(x^*) + \nabla f(x^*)\cdot(x-x^*) + h(x)(x-x^*)$
and $h(x)\to 0$ as $x\to x^*$. As $f(x) - f(x^*) \leq 0$

$$\Rightarrow \quad 0 \geq \nabla f(x^*)(x-x^*) + h(x)\cdot(x-x^*) \quad / \cdot \|x-x^*\|$$

$$0 \geq \nabla f(x^*)\frac{(x-x^*)}{\|x-x^*\|} + h(x)\frac{x-x^*}{\|x-x^*\|}$$



unit vector

$$\Rightarrow \forall \vec{e} \quad \nabla f(x^*)\cdot e \geq 0$$

$$\Rightarrow \quad \nabla f(x^*)\cdot e = 0$$

$$\Rightarrow \quad \nabla f(x^*) = 0.$$

Ω

2)

$$f(x) = f(x^*) + \overset{0}{\overset{\parallel}{\nabla f(x^*)}}\cdot(x-x^*) + \frac{1}{2}(x-x^*)\, D^2 f(x^*)\,(x-x^*) + h(x)\,\|x-x^*\|^2$$

Then $0 \geq \vec{e}\cdot D^2 f(x^*)\,\vec{e} \quad \forall \vec{e}$ unit vectors. $\square$

3) $f_n, f : \Omega \to \mathbb{R}$, Ω bdd, $f_n \rightrightarrows f$.

Claim: $\sup_A f_n \to \sup_A f$ for all A compact $A \subset \Omega$!

(up to a subsequence)

<u>Proof</u>: Let x_n be s.t. $f_n(x_n) = \sup_{x\in A} f_n(x)$. As $\{x_n\}$ is bdd, $x_{n_k} \to x^* \in A$. Note that $f_{n_k}(x_{n_k}) \to f(x^*)$ as

$$|f_{n_k}(x_{n_k}) - f(x^*)| \leq \underbrace{|f_{n_k}(x_{n_k}) - f(x_{n_k})|}_{\leq\, \|f_{n_k} - f\|_\infty} + |f(x_{n_k}) - f(x^*)|$$

Hence, $f(x^*) = \lim_{n_k\to\infty} f_{n_k}(x_{n_k}) \geq \lim_{n_k\to\infty} f_{n_k}(x) = f(x)$

$\Rightarrow f(x^*) = \sup_A f(x).$

Problem 3

Suppose first that $u_t - \Delta u < 0$ in $(0,T] \times \Omega$.
Then $\Delta u \leq 0$. Moreover $u_t \geq 0$ (not nec. 0 as it may happen for $t=T$). This contradicts $u_t - \Delta u < 0$ so the max is attained on the parabolic boundary.
In the general case we consider $v = u + \varepsilon |x|^2$

$$v^\varepsilon_t = u_t \quad \Delta v^\varepsilon = \Delta u + 2\varepsilon$$

$$v^\varepsilon_t - \Delta v^\varepsilon = u_t - \Delta u - 2\varepsilon \leq -2\varepsilon < 0 \quad \forall_{\varepsilon > 0}.$$

Then $\sup_{x \in \bar\Omega \times [0,T]} v^\varepsilon = \sup_{x \in \partial\Omega \text{ or } t=0} v^\varepsilon$. By Problem 2 we conclude the proof.

Problem 4

Consider difference $u = u_1 - u_2$ solving

$$\begin{cases} u_t - \Delta u = 0 & (0,T) \times \Omega \\ u(0,x) = 0 & x \in \Omega \\ u(t,x) = 0 & x \in \partial\Omega \end{cases}$$

From Problem 3 we have $u_1 = u_2$.

Problem 5

$$\begin{cases} u_t - \Delta u = f^1 \\ u(0,x) = u_0(x) & x \in \Omega \\ u(t,x) = g^1(x) & x \in \partial\Omega \end{cases} \qquad \begin{cases} v_t - \Delta v = f^2 \\ v(0,x) = v_0(x) \\ v(t,x) = g^2(x) \end{cases}$$

Suppose that $f^1 \leq f^2$, $g^1 \leq g^2$ and $u_0 \leq v_0$. Then $u \leq v$.

<u>Proof:</u> consider $\tilde{u} = u - v$. Then

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = f^1 - f^2 \leq 0 \\ \tilde{u}(0,x) = u_0 - v_0 \leq 0 \\ \tilde{u}(t,x) = g^1 - g^2 \leq 0 \end{cases}.$$

From Problem 3 $\tilde{u} \leq 0 \Rightarrow u \leq v$. $\square$.

Problem 7

Let $u := u_1 - u_2$. Then

$$\begin{cases} u_t - \Delta u = 0 \\ u(0,x) = 0 \\ \frac{\partial u}{\partial n} = 0 \end{cases} \qquad \frac{d}{dt} \int_\Omega |u|^2 dx = 2 \int_\Omega u u_t =$$

$$= 2 \int_\Omega u \Delta u = -2 \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u \overset{=0}{\frac{\partial u}{\partial n}} = -2 \int_\Omega |\nabla u|^2 \leq 0.$$

$\square$.

Problem 9

We define $u(t) = u_0 e^{At}$ where

$$e^{At} = \sum_{k \geq 0} \frac{(At)^k}{k!}$$ convergent in the Banach space $\mathcal{L}(X, X)$. As in the course of ODEs, one checks that $\frac{d}{dt} u(t) = A u(t)$.

However, this is no longer true if A is unbounded.