Problem Set B1.

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7 $\Omega \subset \mathbb{R}^n$

If $V \subset \Omega$ is compact, $C^l(V)$ is a Banach space with a natural norm:

$$\|u\|_{C^l(V)} = \sum_{k=0}^{l} \|D^k u\|_\infty, \qquad \|D^k u\|_\infty = \sup_{x \in V} |D^k u(x)|.$$

Let T be a linear functional on $C_c^\infty(\Omega)$.

T is a distribution if for all compact $V \subset \Omega$ there are C, l such that

$$|T(\varphi)| \leq C \, \|\varphi\|_{C^l(V)} \qquad (*)$$

(l, C — may depend on V).

The minimal l s.t. $(*)$ holds for all $V \subset \Omega$ is called the degree of distribution.

Examples :

(A 1) $u \in L^1_{loc}(\Omega)$ defines distribution

$$T_u(\varphi) = \int_\Omega u(x)\,\varphi(x)$$

Fix $V \subset \Omega$, φ s.t. $\operatorname{supp}\varphi \subset V$.

$$|T_u(\varphi)| \leq \int_V u(x)\,\varphi(x)\,dx \leq$$

$$\leq \|u\|_{L^1(V)} \, \|\varphi\|_{C^0(V)}.$$

degree $= 0$

If $T_u(\varphi) = T_v(\varphi) \Rightarrow \int_\Omega (u(x) - v(x))\,\varphi(x) = 0$

$\forall \varphi \Rightarrow u = v$, a.e.

Important: we ALWAYS identify L^1_{loc} function with T_φ.

(A2) Let $\mu \in \mathcal{M}^+(\Omega)$ bounded.

(actually locally bounded is sufficient)

$$T_\mu(\varphi) = \int \varphi(x)\, d\mu(x)$$

Fix $V \subset \Omega$, φ s.t. supp $\varphi \subset V$. Then

$$|T_\mu(\varphi)| \leq \mu(\Omega)\, \|\varphi\|_{C^0(V)}$$

degree = 0.

(A3), (A4) $\Rightarrow$ HW.

(2) Derivative of a distribution. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

Let $T \in \mathcal{D}'(\Omega)$. Then we define

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|}\, T(D^\alpha \varphi)$$

(B1) The formula defines distribution.

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$$

Fix $V \Subset \Omega$, φ s.t. $\operatorname{supp} \varphi \subset V$. Then

$$|(D^\alpha T)(\varphi)| \leq |T(D^\alpha \varphi)| \leq$$

$$\leq C \, \|D^\alpha \varphi\|_{C^l(V)} \leq C \|\varphi\|_{C^{l+|\alpha|}(V)}$$

$\uparrow$ as $T \in \mathcal{D}'(\Omega)$

(B2) Motivation: let $u \in C_c^\infty(\Omega)$, $\Omega \subset \mathbb{R}$ (so we are in 1D). Identify u with T_u. What is $\partial_x T_u$?

By def. $(\partial_x T_u)(\varphi) = (-1) \, T_u(\partial_x \varphi)$

$$= (-1) \int_\Omega u(x) \partial_x \varphi(x) \, dx = \int_\Omega \partial_x u(x) \varphi(x) \, dx$$

$$= T_{\partial_x u}(\varphi)$$

Now, multi-D case.

total number of derivatives

$$(D^\alpha T_u)(\varphi) = (-1)^{|\alpha|} T_u(D^\alpha \varphi)$$

$$= (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha \varphi(x) = \int_\Omega D^\alpha u(x)\, u(x)$$

$$= T_{D^\alpha u}(\varphi).$$

;)

B3 $u(x) = |x|$

We compute its distributional derivative.

$$T_u(\varphi) = \int_{-1}^{1} |x|\, \varphi(x)\, dx$$

Claim: $(\partial_x T_u)(\varphi) = \int_{-1}^{1} \operatorname{sgn} x \; \varphi(x)\, dx$

Proof: $T_u(\partial_x \varphi) = \int_{-1}^{0} |x| \partial_x \varphi(x)\, dx + \int_0^1 |x| \partial_x \varphi(x)\, dx$

$$= -\int_{-1}^{0} x \partial_x \varphi(x)\, dx + \int_0^1 x \partial_x \varphi(x)\, dx =$$

$$= -\int_{-1}^{0} x\,\partial_x \varphi(x)\,dx + \int_0^1 x\,\partial_x \varphi(x)\,dx =$$

$$= \int_{-1}^{0} \partial_x x\, \varphi(x)\,dx - \underbrace{x\varphi(x)\Big|_{-1}^{0}}_{=0}$$

$$-\int_0^1 \partial_x x\, \varphi(x)\,dx + \underbrace{x\varphi(x)\Big|_0^1}_{=0} =$$

$$= -\int_{-1}^{1} \operatorname{sgn} x\, \varphi(x)\,dx.$$

$$\Rightarrow (\partial_x T_u)(\varphi) = -T_u(\partial_x \varphi) = \int_{-1}^{1} \operatorname{sgn} x\, \varphi(x)\,dx.$$

$\square$.

B4 $\Rightarrow \Uparrow$.

(B5)

$x \mapsto \Phi(x)$ "fund. solution to Laplace eq."

$$T_\Phi(\varphi) = \int_{\mathbb{R}^n} \Phi(x)\, \varphi(x)$$

$$(\partial_{x_i} T_\Phi)(\varphi) = -\int_{\mathbb{R}^n} \Phi(x)\, \partial_{x_i} \varphi(x)\, dx$$

$$(\partial_{x_i x_i} T_\Phi)(\varphi) = \int_{\mathbb{R}^n} \Phi(x)\, \partial^2_{x_i} \varphi(x)\, dx$$

$$(\Delta T_\Phi)(\varphi) = \int_{\mathbb{R}^n} \Phi(x)\, \Delta\varphi(x)\, dx$$

From lecture: for all $u \in C^2(\bar{\Omega})$, for all Ω:

$$u(x) = -\int_{\partial\Omega} u(y)\, \underbrace{\frac{\partial \Phi}{\partial \mathbf{n}}}_{=0}(y-x)\, \mathrm{d}S(y) + \int_{\partial\Omega} \Phi(y-x)\, \underbrace{\frac{\partial u}{\partial \mathbf{n}}(y)}_{=0}\, \mathrm{d}S(y) - \int_{\Omega} \Phi(y-x)\, \Delta u(y)\, \mathrm{d}y.$$

$$\Rightarrow (\Delta T_\Phi)(\varphi) = u(0) = \int u(x)\, d\delta_0(x).$$

measure !!!