Problem Set B2.

Kuba Skrzeczkowski

A concept of derivative between strong and distributional derivative is a WEAK DERIVATIVE.

Def. We say that $u \in L^1_{loc}(\Omega)$ is weakly differentiable if $\exists\, v_i \in L^1_{loc}$ $\forall \phi \in C_c^\infty(\Omega)$

$$\int_\Omega \partial_{x_i} \phi \, u = (-1) \int_\Omega \phi \cdot v_i$$

In this case $u_{x_i} := v_i$. This identity uniquely determines v_i.

Def. $W^{1,p}(\Omega)$ = space of $u \in L^p(\Omega)$ s.t. Du exists and belongs to $L^p(\Omega)$.

Remark $W^{1,p}(\Omega)$ is the space of $u \in L^p(\Omega)$ such that their distr. derivative is a function in $L^p(\Omega)$.

Indeed, $T_u(\phi) = \int_\Omega u(x)\phi(x)\,dx$

$$(\partial_{x_i} T_u)(\phi) = (-1)\int_\Omega u(x)\, \partial_{x_i}\phi(x)\,dx$$

If this is a function v_i, it means

$$(-1)\int_\Omega u(x)\,\partial_{x_i}\phi(x)\,dx = \int_\Omega \phi(x)\,v_i(x)\,dx.$$

A1 $|x| \in W^{1,p}(-1,1)$?

We know that its dist. derivative is $\operatorname{sgn} x$. It is in $L^p(-1,1)$.

A2 $\mathbb{1}_{x>0} \in W^{1,p}(-1,1)$?

We know that its distr. derivative is δ_0 (measure)

$$(-1)\int_{(-1,1)} u(x)\,\partial_x\phi(x) = \phi(0)$$

If $\mathbb{1}_{x>0} \in W^{1,p}(-1,1)$, $\exists_v$ s.t.

$$\phi(0) = \int_{(-1,1)} v(x)\,\phi(x) \quad \forall \phi \in C_c^\infty(-1,1).$$

$\Rightarrow v = 0$ a.e. Contradiction.

A4 $u \in C^1(\bar{\Omega}) \Rightarrow u \in W^{1,p}(\Omega) \ \forall p$

(ass. Ω is bounded)

- $u \in L^p(\Omega)$ as u is cont, Ω is bounded.
- We look for v s.t. $v_i \in L^p(\Omega)$ and

$$\int_\Omega v_i \cdot \phi(x)\, dx = (-1) \int_\Omega u(x)\, \partial_{x_i} \phi(x)$$

It follows that $v_i = \partial_{x_i} u$.

A5 $u(x) = |x|^{-\alpha}$ $\alpha > 0$, d, p s.t. $u \in W^{1,p}(B_1(0))$.

we keep $p=1$ for simplicity

except $x=0$ $\partial_{x_i}|x| = \partial_{x_i}\sqrt{|x|^2} = \dfrac{2x_i}{2\sqrt{|x|^2}} =$

$$= \frac{x_i}{|x|}$$

Hence $\partial_{x_i}|x|^{-\alpha} = -\alpha|x|^{-\alpha-1}\dfrac{x_i}{|x|} = -\dfrac{\alpha x_i}{|x|^{\alpha+2}}$

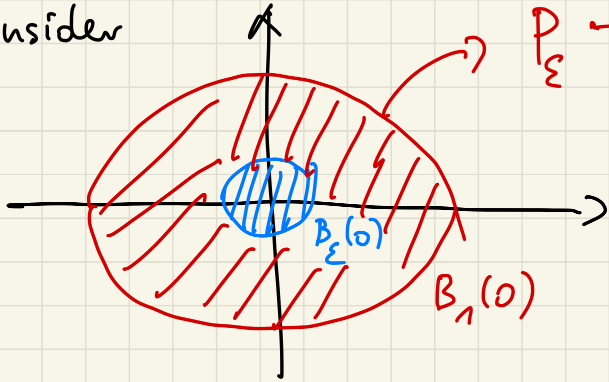
$$|Du| = -\frac{\alpha}{|x|^{\alpha+1}}$$

We want to prove that

$$\forall_{\phi} \quad \int_{B_1(0)} |x|^{-\alpha} \partial_{x_i} \phi(x) = -\int_{B_1(0)} \left(\frac{-\alpha x_i}{|x|^{\alpha+2}} \right) \phi(x)$$

We cannot change variables on $B_1(0)$ as $|x|^{-\alpha}$ is not smooth at $x=0 \in B_1(0)$.

Consider



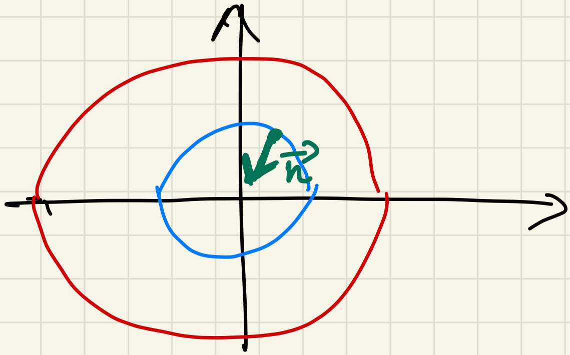
$P_\varepsilon \longrightarrow B_1(0)$ as $\varepsilon \to 0$.

We can integrate by parts in P_ε:

$$\int_{P_\varepsilon} |x|^{-\alpha} \partial_{x_i} \phi(x) = -\int_{P_\varepsilon} \left(-\frac{\alpha x_i}{|x|^{\alpha+2}} \right) \phi(x)$$

$$- \quad + \int_{\partial P_\varepsilon} |x|^{-\alpha} \phi(x)\, n_i(x)\, dS(x)$$

Boundary term does not vanish on the inner boundary.



$$\left| \int_{\partial P_\varepsilon} |x|^{-\alpha} \phi(x)\, n_i(x)\, dS(x) \right| = \left| \int_{\partial P_\varepsilon} \varepsilon^{-\alpha} \phi(x) n_i(x)\, dS(x) \right|$$

$$\leq \|\phi\|_\infty \underbrace{\|n\|}_{=1} \varepsilon^{-\alpha} |P_\varepsilon| \leq C \varepsilon^{-\alpha} \varepsilon^{n-1}$$

If $n-1-\alpha > 0$ i.e. $\alpha+1 < n$ this boundary term converges to 0.

(A) Convergence $\int_{P_\varepsilon} |x|^{-\alpha} \partial_{x_i} \phi(x) \to \int_{B_1(0)} |x|^{-\alpha} \partial_{x_i} \phi(x)$

can be justified with DCT if $\int_{B_1(0)} |x|^{-\alpha} =$

$$= \int_0^1 r^{-\alpha} r^{n-1}\, dr < \infty \text{ if } n > \alpha.$$

(B) Similarly, $\int_{\varepsilon}^{?} \frac{-\alpha x_i}{|x|^{\alpha+2}} \phi(x) \to \int_{B_1(0)} \frac{-\alpha x_i}{|x|^{\alpha+2}} \phi(x)$

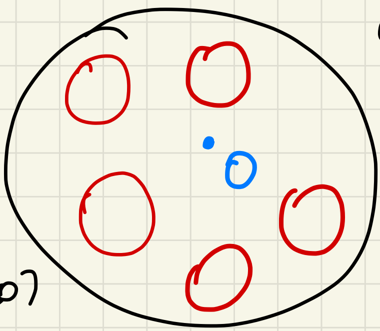
If $\frac{-\alpha x_i}{|x|^{\alpha+2}} \in L^1 \; \forall_i$ i.e. $\frac{1}{|x|^{\alpha+1}} \in L^1(B_1(0))$

and this is for $\boxed{\alpha+1 < n}$

CLAIM : $u \in W^{1,1}(B_1(0)) \iff \alpha+1 < n$

($\Leftarrow$) done

($\Rightarrow$) If $u \in W^{1,1}(B_1(0)) \Rightarrow Du$ exists in a weak sense $\Rightarrow$ in each red ball u is classically differentiable so Du coincides with classical gradient except $x=0$



$B_1(0)$

$\Rightarrow \frac{1}{|x|^{\alpha+1}} \in L^1(B_1(0))$

$\Rightarrow \alpha+1 < n$. $\square$.

For $p \neq 1$ we only check when $\frac{1}{|x|^{\alpha+1}} \in L^p(B_1(0))$.

(for $(\alpha+1)p < n$).

A6 In AS: Sobolev function may be unbounded in one point. In fact, it can be unbounded in <u>every</u> open ball in $B_1(0)$).

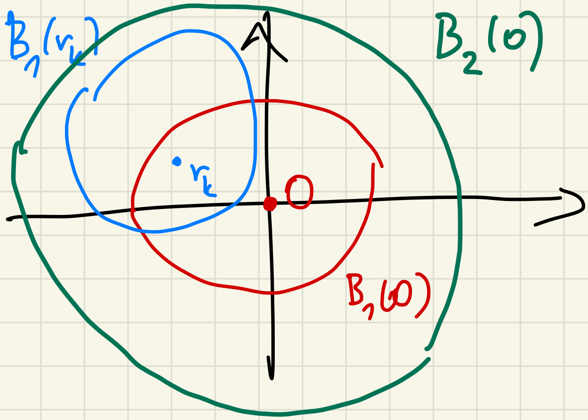
Consider $\{r_k\}_{k\geq 1}$ dense, countable in $B_1(0)$) and let $u(x) = \sum_{k\geq 1} \frac{1}{2^k} |x - r_k|^{-\alpha}$ for α as above

$u \in W^{1,1}(B_1(0))$? $\quad (\alpha + 1 < n)$

(Functional Analysis: $\sum x_k$ converges in BS if $\sum \|x_k\| < \infty$)

- $\sum_k \frac{1}{2^k} \|(x-r_k)^{-\alpha}\|_{L^1(B_1(0))}$
- $\sum_k \frac{1}{2^k} \|D(x-r_k)^{-\alpha}\|_{L^1(B_1(0))}$

$|D(x-r_k)^{-\alpha}| = \frac{\alpha}{|x-r_k|^{\alpha+1}}$ $\qquad |(x-r_k)^{-\alpha}| \leq \frac{1}{|x-r_k|^{\alpha}}$



$$\int_{B_1(0)} \frac{1}{|x-r_k|^{\alpha+1}} = \int_{B_1(r_k)} \frac{1}{|x|^{\alpha+1}} \leq$$

$$\leq \int_{B_2(0)} \frac{1}{|x|^{\alpha+1}} = \int_0^2 r^{-\alpha-1} r^{n-1} \, dr$$

$$= r^{n-(\alpha+1)} \Big|_0^2 = 2^{n-(\alpha+1)}$$

Similarly, $\displaystyle\int_{B_1(0)} \frac{1}{|x-r_k|^{\alpha}} = 2^{n-\alpha}$.

Hence, both series are convergent.

(B1) $U \subset \Omega$, $u \in W^{k,p}(\Omega) \Rightarrow u \in W^{k,p}(U)$.

(B2) $F: \mathbb{R} \to \mathbb{R}$, $F \in C^1$, F just Lipschitz.

$$u \in W^{1,p}(\Omega) \Rightarrow F(u) \in W^{1,p}(\Omega)$$
$$\partial_{x_i} F(u) = F'(u)\, \partial_{x_i} u$$

FACT: $u \in W^{1,p}(\Omega)$ $1 \le p < \infty$. Then, there is a sequence $\{u_k\}_{k \geq 1} \subset C^\infty(\bar{\Omega})$ s.t. $u_k \to u$ in $W^{1,p}(\Omega)$.

TYPICAL APPROXIMATION ARG:

We want to extend some property from a smaller class of functions (where it is "obvious") to a larger one. The main tool is density.

If $u \in W^{1,p}(\Omega) \cap C^\infty(\bar{\Omega})$.

$$\int_\Omega F(u)\, \partial_{x_i} \phi = -\int_\Omega F'(u)\, u_{x_i}\, \phi(x)$$

$F(u) \in W^{1,p}$? Yes, because

- $|F(u)| \leq |F(u)-F(0)| + |F(0)|$

$$\leq |u| + |F(0)| \in L^p,$$

- $|F'(u)\, u_{x_i}| \leq |F'(u)| \cdot |u_{x_i}| \in L^p$ if

$F' \in L^\infty$ $\left(\text{but } |F'| \leq \lim\limits_{h\to 0} \dfrac{F(u+h)-F(u)}{h} \leq |F|_{Lip} \right)$.

Let $u \in W^{1,p}(\Omega)$. Consider $u_k \to u$ in $W^{1,p}$

i.e. $u_k \to u$ in L^p

$\nabla u_k \to \nabla u$ in L^p (i.e. $\partial_{x_i} u_k \to \partial_{x_i} u$)

strong and weak

weak derivative

Choosing subsequence if necessary

$u_k \to u$ in L^p, a.e. in Ω

$\nabla u_k \to \nabla u$ in L^p, a.e. in Ω

$$\int_\Omega F(u_k)\,\partial_{x_i}\phi = -\int_\Omega F'(u_k)\,\partial_{x_i}u_k\,\phi$$

$$\Big\downarrow \text{(A)} \qquad\qquad\qquad\qquad \Big\downarrow \text{(B)}$$

$$\int_\Omega F(u)\,\partial_{x_i}\phi \qquad\qquad -\int_\Omega F'(u)\,\partial_{x_i}u\,\phi$$

PROOF OF (A): $\left|\int_\Omega \left[F(u_k)-F(u)\right]\partial_{x_i}\phi\right| \leq$

$$\leq |F|_{Lip}\int_\Omega |u_k-u|\,\partial_{x_i}\phi \leq$$

$$\leq |F|_{Lip}\,\|u_k-u\|_p\,\|\phi\|_{p'} \longrightarrow 0.$$

PROOF OF (B): $\left|\int_\Omega F'(u_k)\,\partial_{x_i}u_k\,\phi - \dots\right| \leq$

$$\leq \int_\Omega |F'(u_k)|\,|\partial_{x_i}u_k-\partial_{x_i}u|\,\phi \; +$$

$$+ \int_\Omega |F'(u_k)-F'(u)|\,|\partial_{x_i}u|\,\phi$$

The first term is easily controlled with Hölder inequality as above. For the second, we note that

$$F'(u_k) \longrightarrow F'(u) \quad \text{a.e. as } F \in C^1.$$

As

$$\left| \left| F'(u_k) - F'(u) \right| \partial_{x_i} u \, \phi \right| \leq$$

$$\leq 2 \|F'\|_\infty \, |\partial_{x_i} u| \, \phi \in L^1$$

we use dominated convergence to conclude the proof.

$\square$.

B3 $\Uparrow$

C1 $\quad W_0^{1,p} = \overline{C_c^\infty}^{W^{1,p}}$

<u>$1 \le p < \infty$</u>: First, consider $u \in C_c^\infty$.

$$\int_\Omega |u|^p = \int_\Omega \partial_{x_i}(x_i)|u|^p =$$

$$= - \int_\Omega x_i \, |u|^{p-1} \operatorname{sgn} u \, \partial_{x_i} u \lesssim$$

WRONG!

$$\lesssim C(\Omega) \int_\Omega |u|^{p-1} |Du| \lesssim$$

$$\lesssim C(\Omega) \left(\int_\Omega |u|^{(p-1)q} \right)^{1/q} \left(\int_\Omega |Du|^p \right)^{1/p}$$

$$1 = \frac{1}{p} + \frac{1}{q} \Rightarrow \frac{1}{q} = \frac{p-1}{p}, \quad q = \frac{p}{p-1}$$

$$\ldots \lesssim C(\Omega) \, \|u\|_{L^p}^{\frac{p-1}{p}} \, \|Du\|_{L^p}$$

$$\Rightarrow \left(\int_\Omega |u|^p \right)^{1/p} \lesssim C(\Omega) \left(\int_\Omega |Du|^p \right)^{1/p}.$$

To correct this argument we recall function

$$F_\varepsilon(z) = (z^2+\varepsilon^2)^{1/2} - \varepsilon = \frac{z^2}{\sqrt{z^2+\varepsilon^2}+\varepsilon} \leq z.$$

$F_\varepsilon \in C^1$, 1-Lips., $F_\varepsilon \to |z|$, $F_\varepsilon' \to \operatorname{sgn} z \, 1\!\!1_{z\neq 0}$

$$\int_\Omega |F_\varepsilon(u)|^p = \int_\Omega \partial_{x_i}(x_i)\, |F_\varepsilon(u)|^p =$$

$$= - \int_\Omega x_i \, (F_\varepsilon(u))^{p-1} \overbrace{F_\varepsilon'(u)}^{\leq 1} u_{x_i} \leq$$

$$\leq C(\Omega) \int_\Omega (F_\varepsilon(u))^{p-1} |Du| .$$

By Hölder $\|F_\varepsilon(u)\|_{L^p} \leq C(\Omega) \|Du\|_{L^p}$.

As u is smooth, $\|F_\varepsilon(u)\|_{L^p} \to \|u\|_{L^p}$.

(we use here DCT with $|F_\varepsilon(u)| \leq |u|$).

(C2) It also works for unbounded Ω if there exists direction in which Ω is bounded.

(simply, we use this direction in the proof).

(C3) No, we choose $u = 1$.

(C4) $\Uparrow$

IMPORTANT TECHNIQUE: LOOKING FOR CONTINUOUS REPRESENTATIVES.

(C5) $u \in W^{1,p}(0,1)$, $1 < p < \infty$.

1. If u is smooth

$$|u(x) - u(y)| = \left| \int_y^x u'(z)\,dz \right| \le$$

$$\le \left| \int_0^1 u'(z) \cdot \mathbb{1}_{[y,x]} dz \right| \le \quad [\text{Hölder}]$$

$$\le \left(\int_0^1 |u'(z)|^p \right)^{1/p} |x-y|^{1-1/p} \le \|u'\|_p \, |x-y|^{1-1/p}.$$

2. If $u \in W^{1,p}(0,1)$ then we take a sequence

$u_n \in C^\infty([0,1])$ s.t. $u_n \to u$ in $W^{1,p}(0,1)$,
$u_n \to u$ a.e. in $(0,1)$,
$u_n' \to u'$ a.e. in $(0,1)$.

Hence, for a.e. $x, y \in [0,1]$

$$|u(x) - u(y)| \leq \|u'\|_p \, |x-y|^{1-1/p} \quad (*)$$

3. Consider restriction of u to $[\delta, 1-\delta]$

$$u^\varepsilon = u * \eta_\varepsilon. \quad (0 < \varepsilon < \delta/2)$$

We claim that $\{u^\varepsilon\}$ satisfies assumptions of AA theorem:

- $\{u^\varepsilon\}$ is bounded on $[0,1]$:

First, $u \in L^\infty$ by $(*)$ so by Young's ineq.

$$\|u^\varepsilon\|_\infty \leq \|u\|_\infty \|\eta_\varepsilon\|_1 = \|u\|_\infty$$

- $\left|u^\varepsilon(x) - u^\varepsilon(y)\right| = \left|\int \left[u(x-z) - u(y-z)\right] \eta_\varepsilon(z)\,dz\right|$

$$\leq \|u'\|_p \, |x-y|^{1-1/p}$$

Therefore, up to a subsequence $u^\varepsilon \rightrightarrows u$ in $C[\delta, 1-\delta]$ so that $u \in C([\delta, 1-\delta])$.

4. Clearly $u^\varepsilon(x) - u^\varepsilon(y) = \int_x^y (u^\varepsilon)'(z)\,dz$

If $x, y \in (\delta, 1-\delta)$, $u^\varepsilon(x) \to u(x)$
$u^\varepsilon(y) \to u(y)$

Concerning $\int_x^y (u^\varepsilon)'(z)\,dz = \int_x^y (u')^\varepsilon(z)\,dz$

As $u' \in L^p(0,1)$, $(u')^\varepsilon \to (u')$ in $L^p(\delta, 1-\delta)$

$\Rightarrow (u')^\varepsilon \to u'$ in $L^1(\delta, 1-\delta)$. Hence,

$u(x) - u(y) = \int_x^y (u')(z)\,dz$. for $x, y \in (\delta, 1-\delta)$

and so for all $x, y \in (0,1)$.

5. The last part aims at finding continuous extension of u at $x=0$, $x=1$.

We have

$$|u(x)-u(y)| \leq \|u'\|_p |x-y|^{1-1/p}$$

We know that there is <u>unique</u> extension of unif. cont. function from $(0,1)$ to $[0,1]$.

Moreover, taking limits:

$$u(x) - u(y) = \int_0^1 u'(z)\, dz$$

$$|u(x)-u(y)| \leq \|u'\|_p |x-y|^{1-1/p}.$$