Problem Set B3.

Kuba Skrzeczkowski

A1 $\partial_{x_i}(u * \eta_\varepsilon) = u * \partial_{x_i}\eta_\varepsilon$ obvious.

To prove $\partial_{x_i}(u * \eta_\varepsilon) = (\partial_{x_i} u) * \eta$ we consider test function $\psi \in C_c^\infty(\mathbb{R}^n)$

$$\int \partial_{x_i}(u * \eta_\varepsilon)\, \psi(x) = - \int u * \eta_\varepsilon\, \partial_{x_i}\psi$$

$$= - \int u\, ((\partial_{x_i}\psi) * \eta_\varepsilon) = - \int u\, \partial_{x_i}(\underbrace{\psi * \eta_\varepsilon}_{\text{new test function}})$$

$$= \int \partial_{x_i} u\, \psi * \eta_\varepsilon = \int (\partial_{x_i} u) * \eta_\varepsilon\, \psi$$

As ψ is arbitrary $\partial_{x_i}(u * \eta_\varepsilon) = (\partial_{x_i} u) * \eta_\varepsilon$.

$\square$.

A2 Fix $x \in \Omega_\varepsilon$ and consider $u * \eta_\varepsilon$.

Then, with A1, $D(u * \eta_\varepsilon) = (Du) * \eta_\varepsilon = 0$

$\Rightarrow u * \eta_\varepsilon$ is constant.

As $u * \eta_\varepsilon \longrightarrow u \Rightarrow u$ is constant a.e.

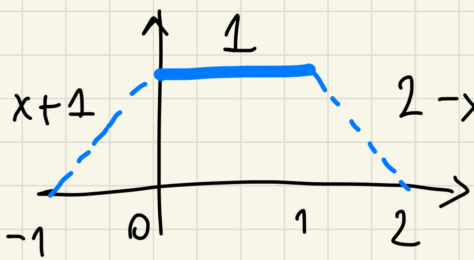
A3 This follows from definition.

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p}(\Omega)}$$

$$W^{1,p}(\Omega) = \overline{C^\infty(\bar{\Omega})}^{W^{1,p}(\Omega)}$$

B1

$u = \mathbb{1}_{[0,1]} \in W^{1,1}(0,1)$



Candidate

$$u' = \begin{cases} 0 & (-\infty, -1) \\ 1 & (-1, 0) \\ 0 & (0,1) \\ -1 & (1,2) \\ 0 & (2,\infty) \end{cases}$$

$$= \mathbb{1}_{(-1,0)} - \mathbb{1}_{(1,2)}$$

We need to check $\int u \varphi' = -\int u' \varphi$.

$$\int u\varphi' = \int_{-1}^{2} u\varphi' = \left(\int_{-1}^{0} + \int_0^1 + \int_1^2\right) u\varphi'$$

$$= -\int_{-1}^{0} \varphi + \underbrace{(x+1)\varphi \Big|_{-1}^{0}}_{= \varphi(0)} + \underbrace{\int_0^1 \varphi'}_{= \varphi(1) - \varphi(0)}$$

$$- \int_1^2 (-1)\,\varphi + \underbrace{(2-x)\varphi \Big|_1^2}_{= \varphi(1)} =$$

$$= -\int_{\mathbb{R}} \varphi \left[\mathbb{1}_{[-1,0)} - \mathbb{1}_{(1,2)} \right]. \qquad \square.$$

B2 For $u \in W_0^{1,p}$, the trivial extension works

$$\Uparrow$$

B3 Similar extension results work for $C^k(\bar{\Omega})$ (Whitney, Annals of Mathematics, '30).

C1 For $u \in W_0^{1,p}(\Omega)$ we have Poincare

$$\|u\|_p \leq C \|Du\|_p.$$

Hence $1 \notin W^{1,p}(\Omega)$.

C2 $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

If $u \in L^p$, $\exists_{(u_n) \subset C_c^\infty(\Omega)}$ $u_n \to u$ in $L^p(\Omega)$.

$Tu_n = 0$ $\quad \|Tu\| \leq \underbrace{\|T(u_n - u)\|}_{\to 0} + \underbrace{\|Tu_n\|}_{=0}$

$\Rightarrow Tu = 0 \quad \forall_u$.

But for $u = 1$ $\quad Tu = 1$ contradiction.

$\square$.

C3 $1 < p < \infty$

$$\varphi(u) = u(0) \in \left(W^{1,p}(0,1)\right)^*$$

$$\exists_C \quad |u(0)| \leq C \|u\|_{W^{1,p}(0,1)}. \quad ?$$

$$|u(0)| \leq |u(x)| + \|u\|_{p'} \underbrace{|x|^{1-1/p}}_{\leq 1}$$

$$\leq |u(x)| + \|u\|_{p'}$$

Integrating artificially in x we deduce

$$|u(0)| = \int_0^1 |u(0)|\,dx \leq$$

$$\leq \int_0^1 |u(x)|\,dx + \|u'\|_p \leq \text{(Hölder}$$

$$\leq \|u\|_p + \|u'\|_p = \|u\|_{W^{1,p}(0,1)}. \qquad \square.$$

C4 $\Uparrow$

E1 We take a seq. $\{u_n\}_{n\geq 1}$ bdd in $W^{1,1}(I)$.
We want to prove it has a subseq. converging a.e.

(A) By Extension Theorem, $\{u_n\}$ can be extended to $W^{1,1}(\mathbb{R}^+)$, $\|u_n\|_{W^{1,1}(\mathbb{R}^+)} \leq C\|u_n\|_{W^{1,1}(I)}$ and $\operatorname{supp} u_n \subset J$. ($J$ is some larger, yet still bounded interval).

(B) $u_n^\varepsilon := u_n * \eta_\varepsilon$

Claim: $u_n^\varepsilon \longrightarrow u_n$ $(\varepsilon \to 0)$, unif. in n.

Proof:

$$u_n^\varepsilon(x) - u_n(x) = \int_{B(0,\varepsilon)} [u_n(x-y) - u_n(x)]\eta^\varepsilon(y)\,dy$$

$$\underset{z=\frac{y}{\varepsilon}}{=} \int_{B(0,1)} \underbrace{\left(u_n(x-\varepsilon z) - u_n(x)\right)}_{\text{we estimate this}} \eta(z)\,dz$$

If u_n is smooth,

$$u_n(x-\varepsilon z) - u_n(x) = \int_0^1 \frac{d}{dt} u_n(x-\varepsilon t z)\, dt$$

$$= \int_0^1 u_n'(x-\varepsilon t z)\,(-\varepsilon z)\, dt$$

As $|z| \leq 1$,

$$|u_n(x-\varepsilon z) - u_n(x)| \leq \varepsilon \int_0^1 |u_n'(x-\varepsilon t z)|\, dt$$

If $u_n \in W^{1,1}(J)$, the inequality is satisfied by approximation for a.e. x. This is enough as we will integrate in x. Coming back

$$|u_n^\varepsilon(x) - u_n(x)| \leq \varepsilon \|\eta\|_\infty \int_{B(0,1)} \int_0^1 |u_n'(x-\varepsilon t z)|\, dt\, dz$$

$$\|u_n^\varepsilon - u_n\|_{L^1(J)} \leq \varepsilon \|\eta\|_\infty \int_J \int_{B(0,1)} \int_0^1 \cdots$$

$$= \varepsilon \|\eta\|_\infty \int_{B(0,1)} \int_0^1 \int_J |u_n'(x-\varepsilon t z)|\, dx\, dt\, dz$$

Fubini, nonneg. integrand

[translation]

$$\leq \varepsilon \|\eta\|_\infty \int_{B(0,1)} \int_0^1 \|u_n'\|_{L^1(J)} \, dt \, dz$$

$$\leq C\varepsilon \underbrace{\sup_n \|u_n'\|_{L^1(J)}}_{\leq \sup_n \|u_n'\|_{W^{1,1}(J)} \lesssim C < \infty.}$$

(C) Fix $\varepsilon > 0$. We need to prove

$\Rightarrow \|u_n^\varepsilon\|_\infty \leq C(\varepsilon)$

By Young's ineq. $\|u_n^\varepsilon\|_\infty \lesssim \|u_n\|_{L^1} \|\eta_\varepsilon\|_\infty$

$$\lesssim \frac{C}{\varepsilon} \|u_n\|_{L^1} \lesssim \tilde{C}/\varepsilon.$$

$\Rightarrow \|Du_n^\varepsilon\|_\infty \lesssim C(\varepsilon).$

Note $Du_n^\varepsilon = Du_n * \eta_\varepsilon$ so again

$$\|Du_n^\varepsilon\|_\infty \leq \|Du_n\|_{L^1} \|\eta_\varepsilon\|_\infty \lesssim C/\varepsilon.$$

(D) Fix $\delta > 0$. There is subseq. $\{u_{n_k}\}$ s.t.

$$\limsup_{n_k, n_l \to \infty} \|u_{n_k} - u_{n_l}\|_{L^1(J)} \leq \delta.$$

Choose $\varepsilon > 0$ so that $\|u_n^\varepsilon - u_n\| \leq \delta/2 \ \forall n$.

Choose subseq. $\{u^\varepsilon_{n_k}\}$ s.t. $u^\varepsilon_{n_k} \rightrightarrows \tilde{U}$ by Arzela-Ascoli.

$$\|u_{n_k} - u_{n_l}\| \leq \|u_{n_k} - u^\varepsilon_{n_k}\| + \|u^\varepsilon_{n_k} - u^\varepsilon_{n_l}\| + \|u^\varepsilon_{n_l} - u_{n_l}\|$$

$$\Rightarrow \limsup_{n_k, n_l \to \infty} \|u_{n_k} - u_{n_l}\| \leq \delta/2 + \delta/2 + \underbrace{\limsup_{n_k, n_l \to \infty} \|u^\varepsilon_{n_k} - u^\varepsilon_{n_l}\|_{L^1(J)}}$$

$$\leq C \limsup_{n_k, n_l \to \infty} \|u^\varepsilon_{n_k} - u^\varepsilon_{n_l}\|_{L^\infty(J)} = 0$$

us this is Cauchy sequence.

(E) We apply diagonal argument to find a subsequence $\{u_{n_k}\}$ s.t.

$$\limsup_{n_k, n_\ell \to \infty} \|u_{n_\ell} - u_{n_k}\|_{L^1(J)} \le 0.$$

Then, $\{u_{n_k}\}$ is convergent as $L^1(J)$ is a Banach space.

<u>Diagonal argument:</u> use (D) with $\delta = 1$ to get subsequence $\{u^{(1)}\}$. From this subsequence, choose a further one $\{u^{(2)}\} \subset \{u^{(1)}\}$ with $\delta = 1/2$.

By induction, we get

$$\underbrace{\{u^{(1)}\}}_{\delta=1} \supset \underbrace{\{u^{(2)}\}}_{\delta=1/2} \supset \dots \supset \underbrace{\{u^{(n)}\}}_{\delta=1/n} \supset \dots$$

Let $u_k = u_k^{(k)}$. Then $\limsup_{n_k, n_\ell \to \infty} \|u_{n_k} - u_{n_\ell}\| \le \frac{1}{N}$ if $n_k, n_\ell \ge N$. As they are arb. big we conclude the proof.

(E3) $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$

i.e. if $\|u_n\|_{W^{1,p}} \leq C$, we can choose a subseq. $u_{n_k} \to u$ in $L^p(\Omega)$.

(A) Consider $p < n$. Then $W^{1,p} \subset\subset L^q$ for $\forall q < p^*$

Note that $p < p^* = \frac{np}{n-p}$ so we may choose $q = p$.

(B) Consider $p = n$. Assume $\|u_k\|_{W^{1,n}} \leq C$.

$$\Rightarrow \|u_k\|_{W^{1,r}} \leq C \quad \forall_{r<n}$$

$$\Rightarrow u_{k_l} \to u \quad \text{in } L^s \quad \forall_{s < r^* = \frac{nr}{n-r}}$$

Choosing r so that $n < r^*$ (possible as $r^* \to \infty$ as $r \to n$), we conclude the proof.

(C) Consider $p > n$. Then $\|u_n\|_{C^{0,\alpha}} \leq C$.

Hence, $\exists u_{n_k} \to u$ uniformly (by A-A). $\square$

(E4) We know from E1 that

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega). \quad \left(\begin{array}{c}\Omega \text{ has} \\ C^1 \text{ boundary}\end{array}\right)$$

As Ω is bounded, $\Omega \subset B$, B is some ball.

Let $\{u_n\} \subset W_0^{1,p}(\Omega)$. Using homework, we extend $\{u_n\} \subset W^{1,p}(B)$. As B has smooth bound $\exists\, u_{n_k} \longrightarrow u$ in $L^p(B) \Rightarrow u_{n_k} \longrightarrow u$ in $L^p(\Omega)$

as $\Omega \subset B$.

(E5) This says that identity operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ is compact.

(E6) Arzela-Ascoli says that if $\{u_n\}_n$ satisfies $\|u_n\|_\infty \leq C$, $\|Du_n\|_\infty \leq C$ then $\exists u_{n_k} \to u$ in $L^\infty(\Omega)$ (even $C(\Omega)$).

R-K may be viewed as an L^p-version of A-A: if $\|u_n\|_p \leq C$, $\|Du_n\|_p \leq C$ then $\exists u_{n_k} \to u$ in $L^p(\Omega)$

(E7) $\exists_{C} \; \forall_{u \in W^{1,p}} \,.\, \|u - (u)_\Omega\|_{L^p} > n \|Du\|_{L^p}$.

Suppose $\forall_n \; \exists_{u_n \in W^{1,p}} \; \|\underbrace{u_n - (u_n)_\Omega}_{v_n}\|_{L^p} > n \|\underbrace{Du_n}_{Dv_n}\|_{L^p}$

$\forall_n \; \exists_{v_n \in W^{1,p}} \; \|v_n\|_p > n \cdot \|Dv_n\|$

$w_n := \frac{v_n}{\|v_n\|_p}$ $\qquad 1 > n \|Dw_n\|_p$, $1 = \|w_n\|_p$.

$0 = (w_n)$

There is subsequence converging in L^p.

i.e. $w_n \longrightarrow w$ in L^p.

Moreover $\|Dw_n\|_p \leq \frac{1}{n} \longrightarrow 0$. But

$$\underbrace{\int w_n \cdot \phi_{x_i}}_{\longrightarrow \int w \phi_{x_i}} = -1 \underbrace{\int \partial_{x_i}(w_n) \cdot \phi}_{\longrightarrow 0}$$

$$\Rightarrow \int w \phi_{x_i} = 0 \quad \forall_{i=1,\dots,n} \Rightarrow \quad w \text{ is constant}$$

As $(w) = 0 \Rightarrow w = 0$. Contradiction

with $\|w\|_{L^p} = 1$.

$\square$,

E8 – E11 $\Uparrow$.