Problem Set C1.

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1. $(A) \Rightarrow (B)$
$(B) \Rightarrow (C)$

This is just integration by parts and using compact supp. or boundary value.

For example

$$\int_\Omega \Delta u \cdot \varphi + \int_\Omega \nabla u \cdot \nabla \varphi = \int_{\partial\Omega} \frac{\partial u}{\partial n} \varphi$$

2. Existence for (C): the widest class.

Uniqueness for (A): the smaller class,

In PDEs we aim at finding some balance between the first and the second.

3. $\underline{(A) \Rightarrow (A^*)}$

If $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$

then trace of u is 0. Then $u \in H^2(\Omega) \cap H_0^1(\Omega)$

$-\Delta u = f$ a.e. follows from $-\Delta u = f$ everywhere

$(B) \Rightarrow (B^*)$ Similar.

4 $(B^*) \Leftrightarrow (B^{**})$

(B^*) $u \in H_0^1(\Omega)$, $\int Du \cdot D\varphi = \int f\varphi \quad \forall \varphi \in C_c^\infty$

(B^{**}) $u \in H_0^1(\Omega)$, $\int \nabla u \cdot \nabla\varphi = \int f\varphi \quad \forall \varphi \in H_0^1(\Omega)$.

If $\varphi \in C_c^\infty \Rightarrow \varphi \in H_0^1(\Omega)$ so $B^{**} \Rightarrow B^*$.

If $\varphi \in H_0^1(\Omega)$, there is $\varphi_n \to \varphi$ in $H^1(\Omega)$.

In particular, $\int \nabla u \cdot \nabla \varphi_n \to \int \nabla u \cdot \nabla \varphi$

$$\int f \varphi_n \to \int f \varphi.$$

5 $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $\int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega f\varphi \quad \forall \varphi \in C_c^\infty$

But $\nabla u \in H^1(\Omega)$, $\nabla\varphi \in C_c^\infty$. Using def. of weak der $-\int_\Omega \Delta u \cdot \varphi = \int_\Omega f\varphi$ i.e. $-\Delta u = f$ a.e.

6. We simply integrate by parts to get

$$\int \nabla u \cdot \nabla \varphi + \int c(x)\, u \cdot \varphi + \int b(x) \cdot \nabla u\, \varphi = 0.$$