# Problem Set C.2.

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(A) $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and

$$\int \nabla u \cdot \nabla \varphi = \int \varphi f$$

$$\parallel$$

$$-\int \Delta u \cdot \varphi$$

Moreover, $u = 0$ on $\partial\Omega \Rightarrow u \in H_0^1(\Omega)$.

(B) $H^1(\Omega)$ is equipped with norm:

$$u \mapsto \left( \|u\|_{L^2}^2 + \|Du\|_{L^2}^2 \right)^{1/2}$$

scalar product:

$$(u, v) = \int u \cdot v + \int Du \cdot Dv$$

This is scalar product

(1) Linearity clear

(2) Conjugate symmetry — no issue, we work on $\mathbb{R}$.

(3) Positive definiteness $\int u^2 + \int |Du|^2 = 0$

$\Rightarrow u$ is $0$.

$H^1_0(\Omega) = \overline{C^\infty_0(\Omega)}^{H^1(\Omega)}$ is closed subspace of BS so it is itself Banach space and with the inherited inner product, it is Hilbert space :).

<u>Note:</u> On $H^1_0(\Omega)$ we have equivalent norm $u \mapsto \|Du\|_2$ by Poincare inequality

$$\|u\|_2 \leq C(\Omega) \|Du\|_2 .$$

(c) We need to find constants

$$|a(u,v)| \leqslant C_1 \|u\| \, \|v\| \leftarrow \|Du\|_2$$

$$a(u,u) \geqslant c\|u\|^2$$

The first one: $\left| \int Du \cdot Dv \right| \leq C \|Du\|_2 \|Dv\|_2$ by Hölder.

The second is obvious as $a(u,u) = \|Du\|_2^2$.

(d) We need to find $C$ s.t.

$$\left| \int_\Omega f\varphi \right| \leq C \|\varphi\|$$

We have, by Hölder, $\left| \int_\Omega f\varphi \right| \leq \|f\|_2 \|\varphi\|_2$

$$\leq \|f\|_\infty \|D\varphi\|_2 .$$

norm on $H_0^1(\Omega)$

Poincare
Hölder again

(e) Problem is equivalent with L-M which gives the unique solution in $H_0^1(\Omega)$.

(f) This is usual approx. argument as

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1}.$$

$2, 3 \Rightarrow \Uparrow$.

(4)

(A) If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and

$$-\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

we multiply eq. by $\varphi \in C^\infty(\bar{\Omega})$ to get

$$-\int \Delta u \cdot \varphi = \int f \varphi$$

$$\| $$

$$\int_\Omega \nabla u \cdot \nabla \varphi - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \varphi}_{=0} \Rightarrow \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \varphi$$

Then, we consider seq. $\varphi_n \to \varphi$, $\{\varphi_n\} \subset C^\infty(\bar{\Omega})$ where $\varphi \in H^1(\Omega)$ is arbitrary.

(B) If there is a weak solution then we take $\varphi = 1$ to get $\int_\Omega f = 0$.

(C) Consider

$$\mathcal{H} = \left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\}.$$

$\mathcal{H}$ is closed in $H^1(\Omega)$ as if $u_n \to u$ in $H^1$, $\{u_n\} \subset \mathcal{H}$ then $u_n \to u$ in $L^2 \Rightarrow u_n \to u$ in $L^1$ and finally $\int_\Omega u_n \to \int_\Omega u$. But $\int_\Omega u_n = 0$.

Therefore $(\mathcal{H}, \|\cdot\|_{H^1})$ is a Banach space and so, it is a Hilbert space. We claim that Lax-Milgram assumptions are satisfied in $\mathcal{H}$.

Indeed, we take $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$

$$\ell(v) = \int_\Omega f \cdot v$$

and we only need to prove coercivity of $a$.

We have $a(u,u) = \int_\Omega |\nabla u|^2 \geq C \|u - (u)_\Omega\|^2$

$\shortparallel$

$u$ as $(u)_\Omega = 0$

$$\left.\begin{array}{l} C\|u\|_{L^2}^2 \leq a(u,u) \\ C\|\nabla u\|_{L^2}^2 = C a(u,u) \end{array}\right\} \Rightarrow \frac{C}{C+1}\|u\|_{H^1}^2 \leq a(u,u)$$

D If $u$ is a weak solution, then $u+1$ is also a weak solution.

5
$$\int \nabla u_1 \cdot \nabla \phi = \int f_1 \cdot \phi \qquad \forall \phi \in H_0^1(\Omega)$$
$$- \int \nabla u_2 \cdot \nabla \phi = \int f_2 \phi \qquad \forall \phi \in H_0^1(\Omega)$$

$$\int \nabla (u_1 - u_2) \cdot \nabla \phi = \int \phi (f_1 - f_2)$$

Take $\phi = u_1 - u_2$

$$\int |\nabla (u_1 - u_2)|^2 = \int (f_1 - f_2)(u_1 - u_2)$$
$$\|$$
$$\|u_1 - u_2\|_{H_0^1}^2$$

Recall $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$. Apply this with $\varepsilon a, \frac{b}{\varepsilon}$

$$ab \leq \frac{\varepsilon^2}{2} a + \frac{b^2}{2\varepsilon^2}.$$

Then $\int (f_1 - f_2)(u_1 - u_2) \leq$

$$\leq \frac{1}{2\varepsilon^2} \| f_1 - f_2 \|^2_{L^2} + \frac{\varepsilon^2}{2} \| u_1 - u_2 \|^2_{L^2}$$

$$\underset{\text{Poincare}}{\leq} \frac{1}{2\varepsilon^2} \| f_1 - f_2 \|^2_{L^2} + \frac{C\varepsilon^2}{2} \| Du_1 - Du_2 \|^2_{L^2}$$

Choose $\varepsilon$ so that $\frac{C\varepsilon^2}{2} < \frac{1}{2}$. Then

$$\frac{1}{2} \| Du_1 - Du_2 \|^2_{L^2} \leq \frac{1}{2\varepsilon^2} \| f_1 - f_2 \|^2_{L^2}. \qquad \square$$

6 $\quad H^{-1}(\Omega) = \left( H^1_0(\Omega) \right)^*$

(A) $f \in L^2(\Omega)$, then $\varphi \mapsto \int f \varphi \in H^{-1}(\Omega)$

Indeed,

$$\left| \int f \varphi \right| \leq \| f \|_{L^2} \| \varphi \|_{L^2} \leq \| f \|_{L^2} \| \varphi \|_{H^1(\Omega)}.$$

(B) $W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \forall q < p^* = \dfrac{pn}{n-p}$

In our case $p=2$ $\quad H^1(\Omega) \subset\subset L^q \quad \forall q < \dfrac{2n}{n-2}$.

and $\|\varphi\|_{L^q} \leq C \|\varphi\|_{H^1(\Omega)}$.

$$\left| \int f \varphi \right| \leq \|f\|_{L^{q'}} \|\varphi\|_{L^q} \leq$$

$$\leq C \|f\|_{L^{q'}} \|\varphi\|_{H^1(\Omega)}$$

$$q < \frac{2n}{n-2} \iff \frac{1}{q} > \frac{n-2}{2n} \iff 1 - \frac{1}{q'} > \frac{n-2}{2n}$$

$$\iff 1 - \frac{n-2}{2n} > \frac{1}{q'} \iff \frac{2n-n+2}{2n} > \frac{1}{q'} \iff$$

$$\iff q' > \frac{2n}{n+2}$$

$n=3 \qquad \dfrac{2n}{n+2} = \dfrac{6}{5} < 2 \quad$ :)

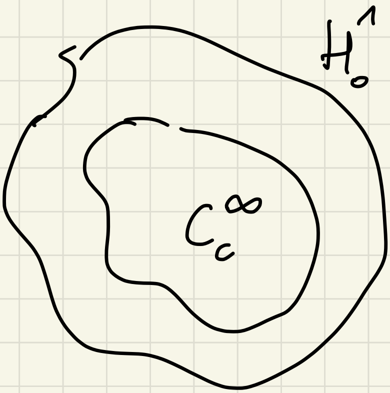
as $n \to \infty \qquad \dfrac{2n}{n+2} \to 2$.

(C) If $f \in L^2(\Omega)$, $\partial_{x_i} f$ extends to $H^{-1}(\Omega)$.

First, $\partial_{x_i} f$ is well-defined for $\varphi \in C_c^\infty(\Omega)$:

$$|(\partial_{x_i} f, \varphi)| = \left| \int_\Omega f \varphi_{x_i} \right| \leq \|f\|_{L^2} \|\varphi_{x_i}\|_{L^2}$$

$$\leq \|f\|_{L^2} \|\varphi\|_{H_0^1(\Omega)}.$$




As $C_c^\infty$ is dense in $H_0^1(\Omega)$ $\partial_{x_i} f$ has a unique extension (with Cauchy sequences).

(D) This is straight forward as Lax-Milgram lemma is originally formulated for $l$ in the dual space.

So we can solve eq. like $-\Delta u = \partial_{x_i} f$ where $f$ is only $L^2(\Omega)$!