Introduction to PDEs (SS 20/21)

(special problems)

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Each problem has assigned number of points and deadline. The deadline may be extended (depending on number of submitted solutions). Please submit the solutions using Moodle.

If not stated otherwise, Ω is always a bounded, open, connected and smooth domain in $\mathbb{R}^n$.

1. (2 points, 22.04.2021) **Mean value property implies continuity.**

   (A) Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$. Prove that the convolution $f * g$ is a continuous and bounded function. *Hint:* First, consider f smooth and compactly supported.

   (B) Suppose that $u : \Omega \to \mathbb{R}$ is an integrable function such that for all $x \in \Omega$

   $$u(x) = \fint_{\partial B(x,r)} u(y) \, \mathrm{dS}(y)$$

   for all balls compactly contained in Ω. Prove that u is continuous.

2. (2 points, 22.04.2021) **Weyl's Lemma.**
   Let $u \in L^1(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is weakly harmonic if for all $\varphi \in C_c^\infty(\Omega)$ we have

   $$\int_\Omega \Delta\varphi(x)\, u(x) = 0.$$

   (A) Prove that if $u \in C^2(\Omega)$ and $\Delta u = 0$ then u is weakly harmonic.

   (B) Prove the converse: if u is weakly harmonic then $u \in C^2(\Omega)$ and $\Delta u = 0$. *Hint:* Mollifiers and mean value property.

   This is the simplest example that motivates and illustrates modern approach to PDEs: first, prove existence of some (seemingly) much weaker solution and then upgrade its regularity to the strong solution.

3. (2 points, 27.05.2021) **Difference quotients A.**
Let $u : \Omega \to \mathbb{R}$ and let U be compactly contained in Ω. For $x \in U$ and $h \in \mathbb{R}$ such that $0 < |h| < \text{dist}(U, \partial\Omega)$ we define i-th difference quotient of size h:

$$D_i^h u(x) = \frac{u(x + h\, e_i) - u(x)}{h}$$

where e_i is the usual unit vector. We also define

$$D^h u = (D_1^h u, D_2^h u, ..., D_n^h u).$$

The link between difference quotients and usual derivatives is well-known. The target of this (and the next) problem is to study the link between difference quotients and Sobolev derivatives. As a warm up, use standard approximation argument to prove the following.

Suppose that $1 \le p < \infty$ and $u \in W^{1,p}(\Omega)$. Then

$$\|D^h u\|_{L^p(U)} \le C\, \|u\|_{W^{1,p}(\Omega)},$$

where constant C is independent of h.

4. (3 points, 27.05.2021) **Difference quotients B.**
It is much more interesting to understand when integrability of difference quotients implies that the function has Sobolev regularity. For this we will need Banach-Alaoglu theorem for L^p spaces:

Theorem. Let $1 < p < \infty$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence bounded in $L^p(\Omega)$. Then, $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence converging weakly.
Proof. For $p = 2$ we proved this in the functional analysis class using orthonormal basis and diagonal argument (review this if you don't remember!). For $1 < p < \infty$, one proceeds similarly using separability of $L^p(\Omega)$, its reflexivity and diagonal argument again.

(A) Prove integration by parts formula for difference quotients: if $\varphi \in C_c^\infty(U)$ and h is sufficiently small

$$\int_U u(x)\, D_i^h \varphi(x) = -\int_U D_i^{-h} u(x)\, \varphi(x).$$

(B) Suppose that $1 < p < \infty$, $u \in L^p(\Omega)$ and

$$\|D^h u\|_{L^p(U)} \le C \qquad \text{for } 0 < |h| < \frac{1}{2}\text{dist}(U, \partial\Omega),$$

where C is independent of h. Prove that $u \in W^{1,p}(U)$.

(C) **(important!)** Show with a simple example that (B) cannot be expected for $p = 1$.

5. (3 points, 10.06.2021) **Euler-Lagrange equations and calculus of variations**
This problem is an introduction to the field of calculus of variations that study minimization of functionals of the form

$$I[u] = \int_\Omega F(\nabla u, u, x)\, \mathrm{d}x$$

defined for instance on Sobolev spaces. As an example consider

$$I[u] = \int_\Omega |\nabla u|^2 - f(x)\, u(x)$$

defined for $u \in H^1_0(\Omega)$. Here, f is a fixed function in $L^\infty(\Omega)$ and Ω is a bounded domain. We will see that there exists the unique minimizer of I over $H^1_0(\Omega)$ and it is a weak solution to Poisson equation

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

(A) Prove that there is at most one function $u \in H^1_0(\Omega)$ such that $\inf_{u \in H^1_0(\Omega)} I[u] = I[u]$.

(B) Let $c = \inf_{u \in H^1_0(\Omega)} I[u]$. Prove that c is finite.

(C) Prove that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $I[u_n] \to c$ as $n \to \infty$ and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$.

(D) Use Banach-Alaoglu to obtain a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ converging weakly in $H^1_0(\Omega)$. Prove that its limit u satisfies $I[u] = m$, i.e. u is a minimizer. *Hint:* In Hilbert space H if $x_n \rightharpoonup x$ we have $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

(E) Prove that u solves Poisson equation in the weak sense. *Hint:* Consider $u + \varepsilon\, \phi$ for small ε and arbitrary $\phi \in H^1_0(\Omega)$.

Remark: One can generalize this method to a wider class of functionals. As a consequence, one proves existence and uniqueness to rather complicated elliptic PDEs which could not be attacked directly. The PDE solved by the minimizer is called Euler-Lagrange equation.

6. (4 points, 10.06.2021) **Stampacchia's Theorem**
In this problem we show how to generalize Lax-Milgram Lemma to study nonlinear equations. A particular example we have in mind is

$$\begin{aligned} -\Delta u + g(u) &= f \text{ in } \Omega \subset \mathbb{R}^n \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is bounded, $g : \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz continuous and increasing. I follow the formulation from Problem Set 2 in NPDE I course at UniBonn. To establish existence and uniqueness we prove:

Stampacchia's Theorem. Let H be a Hilbert space. Let $a : H \times H \to \mathbb{R}$. Assume that a satisfies

(1) for each $u \in H$, the map $v \mapsto a(u, v)$ is continuous and linear (it belongs to H^*),
(2) $|a(u_1, v) - a(u_2, v)| \leq \beta \|u_1 - u_2\| \|v\|$,
(3) $a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \geq \gamma \|u_1 - u_2\|^2$

for some constants β and γ. Then for every $l \in H^*$, there exists uniquely determined u such that $a(u, v) = l(v)$ for all $v \in H$.

We proceed as follows:

(A) Prove that if a (nonlinear!) map $A : H \to H$ satisfies
   (1) $\|A(u_1) - A(u_2)\| \leq \beta \|u_1 - u_2\|$,
   (2) $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \gamma \|u_1 - u_2\|^2$,
   then for every $f \in H$ there is a unique $u_f \in H$ such that $A(u_f) = f$.

   Hint: Apply Banach Fixed Point Theorem to the map $R(u) = u - \lambda A(u) + \lambda f$ for appropriate λ.
(B) Prove Stampacchia's Theorem.
(C) Define weak solutions (in $H_0^1(\Omega)$) to (1). Prove that there exists the unique weak solution to (1).