Introduction to PDEs (SS 20/21), Problem Set A1

Transport equation: an example of hyperbolic equations

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1. In this exercise we study the simplest transport equation with constant velocity. Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a C^1 function and $b \in \mathbb{R}$. Consider equation

$$\partial_t u(t,x) + b\,\partial_x u(t,x) = 0, \qquad u(0,x) = u_0(x). \tag{1}$$

   (A) Existence: Prove that if u is a C^1 solution to (1) then $t \mapsto u(t, x+tb)$ is constant. Conclude that $u(t,x) = u_0(x-tb)$ solves equation.

   (B) Uniqueness: Prove that the C^1 solution is uniquely determined with its initial condition.

   (C) Stability: Assume that u_0 is additionally Lipschitz continuous. Let $u^{(1)}$, $u^{(2)}$ be two C^1 solutions to (1) with coefficient $b = b^1$ and b^2 respectively. Prove that

$$|u^{(1)}(t,x) - u^{(2)}(t,x)| \le t\,|b^1 - b^2|.$$

   (D) Stability: Let $u^{(1)}$, $u^{(2)}$ be two solutions to (1) with inital conditions $u_0^{(1)}$ and $u_0^{(2)}$ respectively. Prove that

$$|u^{(1)}(t,x) - u^{(2)}(t,x)| \le \|u_0^{(1)} - u_0^{(2)}\|_\infty.$$

   (E) Maximum principle: Let u be a C^1 solution to (1). Prove that $\|u\|_\infty \le \|u_0\|_\infty$.

   (F) Semigroup property: Let $f \in C^1(\mathbb{R})$. Consider operator $\mathcal{S}_t f$ to be the solution to (1) with initial condition f. Prove that $\mathcal{S}_t \mathcal{S}_s f = \mathcal{S}_{t+s} f$.

   Remark: Points (A) - (D) together are referred to as a well-posedness theory in the sense of Hadamard.

2. Let $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$. Consider equation

$$\partial_t u(t,x) + b \cdot \nabla_x u(t,x) = 0, \qquad u(0,x) = u_0(x). \tag{2}$$

   where $b \in \mathbb{R}^n$. Explain how to adapt results of Problem 1 to (2).

3. Let $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$. Consider equation

$$\partial_t u(t,x) + b \cdot \nabla_x u(t,x) = c u(t,x), \qquad u(0,x) = u_0(x). \tag{3}$$

   where $b \in \mathbb{R}^n, c \in \mathbb{R}$. Explain how to adapt results of Problem 1 to (3). For stability with respect to c you will need $u_0 \in L^\infty$.

4. To generalize further results of Problem 1-3 we need to recall the concept of flow of vector field. Given Lipschitz vector field $b : \mathbb{R}^n \to \mathbb{R}^n$ we define $X_b(t,x)$ to be the solution of

$$\partial_t X_b(t,x) = b(X_b(t,x)), \qquad X_b(0,x) = x.$$

   We say that X_b is an autonomous flow of the vector field b. Prove that:

(A) $X_b(t,x)$ is well-defined for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$.[1]

(B) X_b satisfies the semigroup property: for all $t, s \in \mathbb{R}$

$$X_b(t, X_b(s,x)) = X_b(t+s, x).$$

(C) $x \mapsto X_b(t,x)$ is Lipschitz continuous.

(D) If $b \in C^1$ then $x \mapsto X_b(t,x)$ is C^1.[2]

(E) The inverse of the map $x \mapsto X_b(t,x)$ equals $X_b(-t,x)$.

(F) $x \mapsto X_b^{-1}(t,x)$ is Lipschitz continuous. If $b \in C^1$ then $x \mapsto X_b^{-1}(t,x)$ is also C^1.

5. Hyperbolic equations can be sometimes solved explicitly with the method of characteristics. Consider equation

$$\partial_t u(t,x) + b(x) \cdot \nabla_x u(t,x) = 0, \tag{4}$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$ and $b : \mathbb{R}^n \to \mathbb{R}^n$. Let $X_b(t,x)$ be the flow of the vector field b. Prove that $u(t, X_b(t,x))$ is constant. Conclude that

$$u(t,x) = u_0(X_b^{-1}(t,x)).$$

6. Let $b : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 and Lipschitz vector field. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a C^1 and Lipschitz function. Use Problems 4 and 5 (PS A1) to establish well-posedness theory for C^1 solutions to equation

$$\partial_t u(t,x) + b(x) \cdot \nabla_x u(t,x) = 0, \qquad u(0,x) = u_0(x)$$

similarly as in Problem 1 (PS A1). Comment on existence, uniqueness, stability with respect to vector field b as well as initial data u_0, maximum principle and semigroup property.

7. Explain how to generalize characteristics method from Problem 3 to study equation of the form

$$\partial_t u(t,x) + b(x) \cdot \nabla_x u(t,x) = c(t,x)\, u(t,x). \tag{5}$$

8. Apply characteristics method to find a solution to

$$u_t + x\, u_x = 0, \qquad u_0(x) = \cos x.$$

9. A typical feature of nonlinear hyperbolic equations is formation of discontinuities or singularities i.e. even if one starts with a smooth initial condition, the solution becomes discontinuous in a finite time. As an example, consider Burger's equation

$$u_t + u\, u_x = 0, \qquad u(0,x) = u_0(x),$$

where $u(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$.

(A) Show that characteristics method implies implicit equation $u(t, u_0(x)\, t + x) = u_0(x)$.

(B) Find solution to Burger's equation with $u_0(x) = 1 - x$ for $0 \le t < 1$. What happens at $t = 1$?

[1] Apply well-known fact: if the solution to ODE cannot be extended from the maximal interval of existence, it blows up.

[2] Use differentiability of solutions to ODEs with respect to initial conditions.

10. Sometimes it is physically motivated to consider hyperbolic equations with discontinuous initial conditions. By virtue of characteristics method we know that regularity of solution propagates with time, i.e. if initial condition is discontinuous, there is no hope that the solution becomes smooth. Therefore, we need the concept of solutions which does not use derivatives.

    Let F be Lipschitz continuous. We say that $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ with $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is a distributional solution to the conservation law

$$\partial_t u(t,x) + \partial_x F(u(t,x)) = 0 \tag{6}$$

    if for all $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \partial_t \varphi(t,x)\, u(t,x) + \int_{\mathbb{R}^+ \times \mathbb{R}} \partial_x \varphi(t,x)\, F(u(t,x)) + \int_{\mathbb{R}} u_0(x)\, \varphi(0,x) = 0$$

    Prove that:

    (A) if u is a classical (differentiable) solution then u is a distributional solution,

    (B) distributional solution is well-defined for $u \in L^\infty$ (i.e. all integrals are well-defined),

    (C) if u is C^1 distributional solution then $\partial_t u(t,x) + \partial_x F(u(t,x)) = 0$ (one can also get initial condition but for now it is quite technical).

11. Although distributional solutions are more flexible, they are not unique in general. Consider

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

    Prove that both functions

$$u_1(t,x) = \begin{cases} 0 & \text{if } x < t/2 \\ 1 & \text{if } x > t/2. \end{cases}, \qquad u_2(t,x) = \begin{cases} 1 & \text{if } x > t \\ x/t & \text{if } 0 < x < t. \\ 0 & \text{if } x < 0. \end{cases}$$

    are distributional solutions to (6) with $F(u) = u^2/2$.

12. In many applications (traffic flow, developmental biology, numerical analysis) it is important to study solutions to transport equations in the space of nonnegative measures $\mathcal{M}^+(\mathbb{R})$ equipped with the topology of weak convergence.

    Let $\mu_0 \in \mathcal{M}^+(\mathbb{R})$ and $b \in C^1(\mathbb{R})$. We say that the family of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ solves transport equation

$$\partial_t \mu_t + b(x)\, \partial_x \mu_t = 0 \tag{7}$$

    if for all $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R})$ we have

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \partial_t \varphi(t,x)\, \mathrm{d}\mu_t(x)\mathrm{d}t \;+ \int_{\mathbb{R}^+ \times \mathbb{R}} \partial_x (b\, \varphi(t,x))\, \mathrm{d}\mu_t(x)\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x)\, \mathrm{d}\mu_0(x) = 0$$

    (A) To motivate, suppose that measure μ_t has C^1 density $u(t,x)$ with respect to Lebesgue measure. Prove that u solves classical transport equation.

(B) For $b(x) = b \in \mathbb{R}$ and $\mu_0 = \delta_{x_0}$, compute measure solution to (7).

13. In this exercise we find measure solution to the PDE (called continuity equation)

$$\partial_t \mu_t + \partial_x (b(x)\mu_t) = 0.$$

with initial condition μ_0. Recall that $\{\mu_t\}_{t\in\mathbb{R}^+}$ is a measure solution if for all test functions $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$ we have

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t \varphi(t,x)\,\mathrm{d}\mu_t(x)\mathrm{d}t \;+ \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x \varphi(t,x)\, b(x)\,\mathrm{d}\mu_t(x)\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x)\,\mathrm{d}\mu_0(x) = 0$$

This will generalize the case $b(x) = b$ known from the problem above. The idea is to define appropriate transport operator acting on measures.

(A) Let (X_1, Σ_1), (X_2, Σ_2) be two measure spaces (here Σ_1, Σ_2 denotes σ-algebras of subsets of X_1 and X_2 respectively). Let μ be a nonnegative measure on (X_1, Σ_1) and T be a measurable map $T : (X_1, \Sigma_1) \to (X_2, \Sigma_2)$. Prove that

$$T^{\#}\mu(A) = \mu(T^{-1}(A))$$

defines a nonnegative measure on (X_2, Σ_2). We say that $T^{\#}\mu$ is a push-forward of μ along T.

(B) Let μ be a bounded (i.e. $\mu(X_1) < \infty$) and nonnegative measure. Prove change-of-variables formula: for all bounded $f : X_2 \to \mathbb{R}$ we have

$$\int_{X_2} f(x)\;\mathrm{d}T^{\#}\mu(x) = \int_{X_1} f(T(x))\;\,\mathrm{d}\mu(x).$$

Hint: first, consider simple function $f(x) = \mathbb{1}_A(x)$ where $A \in \Sigma_2$. Then, prove the result for all nonnegative bounded functions and finally for all bounded.

(C) Let μ_0 be a nonnegative bounded measure on $\mathbb{R}$ and let $X_b(t,x) : \mathbb{R} \to \mathbb{R}$ be the flow of the vector field b. Prove that $\mu_t := X_b(t,x)^{\#}\mu_0$ is a measure solution to the continuity equation.