Introduction to PDEs (SS 20/21), Problem Set A2

Laplace and Poisson equation: examples of elliptic equations

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The target is to obtain well-posedness theory for Poisson's equation

$$-\Delta u = f \text{ in } \Omega$$

equipped with Dirichlet boundary conditions $u = g$ on $\partial\Omega$. We will do that for Ω being a ball or the whole space.

Green's identities, integration by parts

Let $u, v \in C^2(\overline{\Omega})$ and $\Omega \subset \mathbb{R}^d$ be a smooth domain for which the divergence theorem holds.

A1. Show that
$$\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}.$$

A2. Prove Green's first identity:
$$\int_\Omega v\, \Delta u \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v = \int_{\partial\Omega} v \frac{\partial u}{\partial \mathbf{n}}.$$

A3. Prove Green's second identity:
$$\int_\Omega (v\, \Delta u - \Delta v\, u) = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}} \right).$$

A4. Prove integration by parts formula
$$\int_\Omega D_j u(x)\, v(x) + \int_\Omega u(x)\, D_j v(x) = \int_{\partial\Omega} u(x)\, v(x)\, n_j(x)$$
where n_j is the j-th component of $\mathbf{n}$.

Properties of harmonic functions

Function $u \in C^2(\Omega)$ is called harmonic (subharmonic, superharmonic) in Ω if it satisfies $\Delta u = 0$ ($\Delta u \geq 0$, $\Delta u \leq 0$) in Ω.

B1. We establish a connection between harmonic and holomorphic functions. In what follows we identify $\mathbb{C}$ with $\mathbb{R}^2$. For $\Omega \subset \mathbb{C}$, we write $\widetilde{\Omega}$ for the subset of $\mathbb{R}^2$ corresponding to Ω.

- (A) Prove that if $u : \Omega \to \mathbb{C}$ is holomorphic then real and imaginary parts of u are harmonic functions as maps from $\widetilde{\Omega}$ to $\mathbb{R}$. *Hint:* use Cauchy-Riemann equations.
- (B) Conversely, assume that Ω is simply connected and let $u : \widetilde{\Omega} \to \mathbb{R}$ be a harmonic function. Prove that there is a holomorphic function $v : \Omega \to \mathbb{C}$ such that real part of v equals u. *Hint:* Consider complex derivative of u, namely $w(x + iy) = u_x(x, y) - i\, u_y(x, y)$ and use path integration to find its antiderivative. Observe that u is the real part of the antiderivative.

B2. Let u be harmonic in Ω. Prove that $v = \varphi(u)$ is subharmonic whenever φ is smooth and convex.

B3. Let u be subharmonic in Ω. Prove that for any ball $B_R(y) \subset \Omega$ we have

$$u(y) \leq \fint_{B_R(y)} u(y), \qquad u(y) \leq \fint_{\partial B_R(y)} u(y).$$

Formulate corresponding results for superharmonic and harmonic functions.

B4. (maximum principle for Laplace equation) Let u be subharmonic in Ω and suppose that u attains its maximum in the interior of Ω. Prove that u is constant. Formulate corresponding results for superharmonic and harmonic functions.

B5. (maximum principle for Laplace equation) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Assume that Ω is bounded. Prove that u attains its supremum on the boundary:

$$\sup_{x \in \Omega} u(x) = \sup_{x \in \partial\Omega} u(x).$$

Formulate corresponding results for superharmonic and harmonic functions.

B6. Find all nonnegative solutions to the nonlinear PDE

$$\Delta u = u^2 \text{ in } B_1(0), \qquad u(x) = 0 \text{ on } \partial B_1(0).$$

Consequences for Poisson's equation

C1. (uniqueness for Poisson's equation) Prove uniqueness for solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to Poisson equation in bounded domains Ω.

C2. (energy method) Use integration by parts to deduce uniqueness for Poisson's equation.

C3. (comparison, maximum principle and stability for Poisson's equation) As always we assume $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ with $\Omega \subset \mathbb{R}^d$ open, connected and bounded. Moreover, we assume that u_1, u_2 solve

$$\begin{cases} -\Delta u = f_1 & \text{in } \Omega \\ u = g_1 & \text{on } \partial\Omega \end{cases}, \qquad \begin{cases} -\Delta v = f_2 & \text{in } \Omega \\ v = g_2 & \text{on } \partial\Omega \end{cases},$$

where $f_1, f_2 \in C(\Omega)$ and $g_1, g_2 \in C(\partial\Omega)$.

(A) (comparison principle) Suppose that $f_1 \leq f_2$, $g_1 \leq g_2$. Then, $u_1 \leq u_2$.

(B) (maximum principle) We have

$$\|u\|_{L^\infty(\Omega)} \leq C \left(\|f_1\|_{L^\infty(\Omega)} + \|g_1\|_{L^\infty(\partial\Omega)} \right),$$

where C is a constant that depends only on the size of Ω. *Hint:* Consider $\widetilde{u}(x) = \frac{u(x)}{\|f_1\|_{L^\infty(\Omega)} + \|g_1\|_{L^\infty(\partial\Omega)}}$ and $w(x) = \frac{M - x_1^2}{2} + 1$ for appropriate M. Apply (A) to w and $\widetilde{u}$.

(C) (stability) Deduce from (B) that

$$\|u - v\|_{L^\infty(\Omega)} \leq C \left(\|f_1 - f_2\|_{L^\infty(\Omega)} + \|g_1 - g_2\|_{L^\infty(\partial\Omega)} \right).$$

C4. Discuss uniqueness for $\Delta u = f$ in $\mathbb{R}^d$.

Green's method and Laplace equation in the ball

We write $\Phi(x)$ for the scaled fundamental solution to Laplace equation

$$\Phi(x) = \begin{cases} \frac{1}{n\,(2-n)\,\alpha_n}\,|x|^{2-n} & \text{if } n > 2, \\ -\frac{1}{2\pi}\log|x| & \text{if } n = 2 \end{cases}$$

D1. Prove estimates

$$|D_i\Phi(x)| \le \frac{1}{n\,\alpha_n}|x|^{1-n}, \qquad |D_{i,j}\Phi(x)| \le \frac{1}{\alpha_n}|x|^{-n}$$

D2. Find fundamental solution (i.e. spherically symmetric function u such that $u''(x) = -\delta_0$) in one space dimension.

D3. Using $\Delta\Phi = -\delta_0$, formally justify that for all $u \in C^2(\overline{\Omega})$

$$u(x) = -\int_{\partial\Omega} u(y)\,\frac{\partial\Phi}{\partial\mathbf{n}}(y-x)\,\mathrm{d}S(y) + \int_{\partial\Omega}\Phi(y-x)\,\frac{\partial u}{\partial\mathbf{n}}(y)\,\mathrm{d}S(y) - \int_{\Omega}\Phi(y-x)\,\Delta u(y)\,\mathrm{d}y$$

(this will be rigorously proved in the lecture).

D4. Prove that if $u \in C^2(\overline{\Omega})$ solves Poisson equation $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$, then

$$u(x) = -\int_{\partial\Omega} g(y)\,\frac{\partial G}{\partial\mathbf{n}}(x,y) + \int_{\Omega} G(x,y)\,f(y)$$

where G is a Green's function for Ω.

D5. Using Green's function for ball (lecture), one obtains representation formula for solution to $\begin{cases} -\Delta u = 0 & \text{in } B_r(0), \\ u = g & \text{on } \partial B_r(0), \end{cases}$ namely:

$$u(x) = -\int_{\partial\Omega} g(y)\,\frac{\partial G}{\partial\mathbf{n}}(x,y) = \frac{r^2-|x|^2}{n\,\omega_n\,r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}.$$

Prove that u defined with this formula is indeed the unique solution to the Laplace equation with boundary data g.
Remark: One can use *Perron's method* to deduce existence on arbitrary domains from existence in the balls. See Gilbarg-Trudinger.

D6. Solvability of Laplace equation in the ball has the following nice consequence. Let $u \in C(\Omega)$. Prove that u is harmonic in Ω if and only if for all balls $B_R(y)$ compactly contained in Ω

$$u(y) = \fint_{\partial B_R(y)} u(x).$$

Hence, mean value property upgrades regularity of u from $C(\Omega)$ to $C^2(\Omega)$. *Hints:* (1) Fix ball $B_R(y)$ and use solvability of Laplace equation to choose harmonic function with appropriate boundary value. (2) Observe that mean value property implies all maximum principle results.

D7. Conclude that the limit of a uniformly convergent sequence of harmonic functions is harmonic.

Existence for Poisson's equation in the balls and the whole space

Given bounded domain Ω, $d \geq 2$ and $f : \Omega \to \mathbb{R}^d$ we define Newtonian potential of f to be

$$w_f(x) := \int_\Omega \Phi(x-y)\, f(y)\, \mathrm{d}y.$$

The plan is to prove $-\Delta w_f = f$.

E1. Prove that there is a smooth function $\xi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \xi \leq 1$, $|\xi'| \leq C$, $\xi(t) = 0$ for $t \leq 1$ and $\xi(t) = 1$ for $t \geq 2$.

E2. Consider $\xi_\varepsilon(x) = \xi(|x|^2/\varepsilon^2)$. Prove that $\xi_\varepsilon(x) = 0$ for $|x| \leq \varepsilon$, $\xi_\varepsilon(x) = 1$ for $|x| \geq \sqrt{2}\,\varepsilon$ and $|\nabla \xi_\varepsilon(x)| \leq C$ for some constant C.

E3. Prove that $\int_\Omega \Phi(x-y)\, \xi_\varepsilon(x-y)\, f(y)\, \mathrm{d}y$ converges uniformly to w_f in Ω.

E4. Let $f \in L^\infty(\Omega)$. Prove that $w_f \in C^1(\mathbb{R}^d)$ and

$$D_i w_f(x) := \int_\Omega D_i \Phi(x-y)\, f(y)\, \mathrm{d}y.$$

Hint: Use ξ_ε to eliminate singularity in $\Phi(x-y)$, namely consider $\int_\Omega \Phi(x-y)\, \eta_\varepsilon(x-y)\, f(y)$.

E5. Let $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$ for some $\alpha \in (0,1]$. Consider Ω_0 such that $\Omega \subset \Omega_0$ and extend $f = 0$ in $\Omega_0 \setminus \Omega$. Then $w_f \in C^2(\Omega)$ and

$$D_{i,j} w_f(x) = \int_{\Omega_0} D_{i,j}\Phi(x-y)\, (f(x) - f(y))\, \mathrm{d}y - f(x) \int_{\partial\Omega_0} D_i \Phi(x-y)\, n_j(y)\, \mathrm{d}S(y)$$

where n_j is the j-th component of $\mathbf{n}$.

Steps:

(A) If $f \in C^\alpha(\Omega)$, then there is a constant C_f such that for all $x, y \in \Omega$,

$$|f(x) - f(y)| \leq C\, |x-y|^\alpha.$$

Use this to prove that the function above (candidate for the second derivative) is well-defined (finite).

(B) As when computing first derivative, use function ξ_ε to remove singularity around $x \approx y$ and consider $v_\varepsilon(x) = \int_\Omega D_i \Phi(x-y)\, \xi_\varepsilon(x-y)\, f(y)\, \mathrm{d}y$. Observe that Ω may be replaced with Ω_0 in the definition of $v_\varepsilon(x)$.

(C) Compute $D_j v_\varepsilon$ and split $f(y) = (f(y) - f(x)) + f(x)$. Use integration by parts in the second term. What can be said about η_ε in the second term?

(D) Prove that $D_j v_\varepsilon$ converges uniformly to the candidate for $D_{i,j} w_f(x)$. Combine this with uniform convergence of v_ε to $D_i w_f(x)$ and with the fact that C^2 is a Banach space to conclude the proof.

E6. Use formula for $D_{i,j} w_f$ to finally conclude $\Delta w_f = -f$ in Ω.

E7. Compare this result with the theorem from the lecture (under assumption $f \in C^2_c(\mathbb{R}^d)$).

E8. Prove that when $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$, there exists the unique $C^2(\Omega)$ solution of Poisson equation with boundary data $g \in C(\partial\Omega)$. Compare again with the case $\Omega = \mathbb{R}^d$.

E9. Discuss how to consider $\Omega = \mathbb{R}^d$ in the construction above.