Introduction to PDEs (SS 20/21), Problem Set A3

Heat equation: an example of parabolic equations

Compiled on 14/04/2021 at 11:55am

We study heat equation

$$u_t - \Delta u = f \text{ in } (0,T) \times \Omega$$

equipped with appropriate boundary and initial conditions (see below for various).

1. (fundamental solution) Recall that the density of standard normal distribution of $\mathcal{N}(0, 2t)$ equals

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

   (A) Prove that u solves heat equation $u_t - \Delta u = 0$ in $(0,\infty) \times \mathbb{R}^n$ (i.e. except $t = 0$).

   (B) Prove that $u(t,x)$ converges weakly to the Dirac mass at $x = 0$ in the probabilistic sense, i.e. for all bounded and continuous $\varphi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} u(t,x)\, \varphi(x) \, \mathrm{d}x \to \int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}\delta_0(x) = \varphi(0) \text{ as } t \to 0.$$

   This illustrates the smoothing effect in parabolic equations: no matter how irregular initial conditions are (they can be singular measures!), the solutions are smooth for $t > 0$.

2. (analysis 1+2) Here is a quick revision of basic analysis we gonna exploit to establish maximum principles.

   (A) Let $u : \Omega \to \mathbb{R}$ and suppose that u attains its maximum in $x^* \in \Omega$. Prove that $\nabla u(x^*) = 0$ and $D^2 u(x^*) \leq 0$ in the sense of matrices.

   (B) Let $\Omega \subset \mathbb{R}^n$ and $A \subset \Omega$ be compact. Suppose that f_n converges uniformly to f on Ω. Prove that, up to a subsequence,

$$\sup_{x \in A} f_n(x) \to \sup_{x \in A} f(x).$$

   The statement in fact holds true without taking subsequences.

3. (maximum principle) We say that u is a subsolution to the heat equation if

$$u_t - \Delta u \leq 0 \text{ in } (0,T] \times \Omega.$$

   Prove that u attains its maximum on the parabolic boundary $t = 0$ or $x \in \partial\Omega$. Formulate corresponding results for solutions and supersolutions.

   Hint: The result follows easily if one considers strict inequality $u_t - \Delta u < 0$. Therefore, consider $v(t,x) = u(t,x) + \varepsilon |x|^2$ and send $\varepsilon \to 0$.

4. (uniqueness) Prove uniqueness for the heat equation in (a bounded, open, smooth, connected) domain Ω:
$$\begin{aligned} u_t - \Delta u &= f \text{ in } (0,T] \times \Omega, \\ u(0,x) &= u_0(x) \text{ for } x \in \Omega, \\ u(t,x) &= g(x) \text{ for } x \in \partial\Omega. \end{aligned} \tag{1}$$

5. (comparison principle) Prove comparison principle for problem (1): if u, v solve
$$\begin{cases} u_t - \Delta u &= f_1 \text{ in } (0,T] \times \Omega, \\ u(0,x) &= u_0(x) \text{ for } x \in \Omega, \\ u(t,x) &= g_1(x) \text{ for } x \in \partial\Omega. \end{cases}, \qquad \begin{cases} v_t - \Delta v &= f_2 \text{ in } (0,T] \times \Omega, \\ v(0,x) &= v_0(x) \text{ for } x \in \Omega, \\ v(t,x) &= g_2(x) \text{ for } x \in \partial\Omega. \end{cases},$$
with $f_1 \le f_2$, $g_1 \le g_2$ and $u_0 \le v_0$ then $u \le v$.

6. (energy method for uniqueness) Let $u \in C([0,T] \times \Omega) \cap C^2((0,T] \times \Omega)$ be a solution to (1).
   (A) Let $f(x) = g(x) = 0$. Define *energy* with $E(t) = \int_\Omega |u(t,x)|^2 \, \mathrm{d}x$. Prove that $E(t)$ is nonincreasing.
   (B) Deduce uniqueness for classical solutions to (1).

   We did a similar thing for Poisson equation.

7. (Neumann boundary condition) A more physically motivated boundary condition is Neumann condition which determines value of $\frac{\partial u}{\partial \mathbf{n}}$ at the boundary $\partial\Omega$. For example, consider problem
$$\begin{aligned} u_t - \Delta u &= f \text{ in } (0,T) \times \Omega, \\ u(0,x) &= u_0(x) \text{ for } x \in \Omega, \\ \frac{\partial u(t,x)}{\partial \mathbf{n}} &= g(x) \text{ for } x \in \partial\Omega. \end{aligned} \tag{2}$$
   Use energy methods to prove uniqueness for this problem.

8. (maximum principle for porous media equation) Let $u(t,x)$ be a smooth functon satisfying
$$u_t - \Delta F(u) \le 0 \text{ in } [0,T] \times \Omega$$
   on the bounded domain Ω. Here, F is a strictly increasing function ($F'(\lambda) > 0$ for all λ). Prove that u attains its maximum either at $t = 0$ or $x \in \partial\Omega$ (i.e. on the so-called parabolic boundary). Deduce uniqueness for porous media equation:
$$\begin{aligned} u_t - \Delta F(u) &= f(x) \text{ in } [0,T] \times \Omega, \\ u(0,x) &= u_0(x) \text{ for } x \in \Omega, \\ u(t,x) &= g(x) \text{ for } x \in \partial\Omega \end{aligned} \tag{3}$$

9. (semigroup theory) Given Banach space X and a bounded linear operator $A : X \to X$ solve equation
$$u_t(t) = Au(t), \qquad u(0) = u_0 \in X.$$
   However, heat equation cannot be solved with this simple device because Δ is not a bounded operator on any sensible Banach space. However, there is a way to approximate Δ with a sequence of bounded operators (Yosida approximation) and the limit of solutions solves heat equation (and many other equations!!!). This is a starting point of the theory of strongly continuous (analytic) semigroups.