Introduction to PDEs (SS 20/21), Problem Set B1

Introduction to distributions and differentiation

Compiled on 21/04/2021 at 6:41pm

Let T be a linear functional on $C_c^\infty(\Omega)$. We say that T is a distribution if for compact $V \subset \Omega$ there exists constants C, l such that

$$|T(\varphi)| \le C(V)\, \|\varphi\|_{C^l(V)}, \text{ (supp } \varphi \subset V).$$

We write $\varphi \in \mathcal{D}'(\Omega)$.
The minimal l that does not depend on V is called a degree of the distribution T.

- A1. Let $u \in L^1_{loc}(\Omega)$. Prove that u defines a distribution $T_u(\varphi) = \int_\Omega u(x)\, \varphi(x)\, \mathrm{d}x$ and find its degree. Prove that the embedding of $L^1_{loc}(\Omega)$ into $\mathcal{D}'(\Omega)$ is injective.
- A2. Let μ be a finite measure on Ω. Prove that μ defines a distribution $T_\mu(\varphi) = \int_\Omega \varphi(x)\, \mathrm{d}\mu(x)$ and find its degree. Prove that the embedding of $\mathcal{M}(\Omega)$ into $\mathcal{D}'(\Omega)$ is injective.
- A3. Let $k \in \mathbb{N}$ and $x_0 \in \Omega$. Prove that $T_k(\varphi) = \varphi^{(k)}(x_0)$ is a distribution and find its degree.
- A4. Prove that the formula $T(\varphi) = \sum_{k=1}^\infty \varphi^{(k)}\,(1/k)$ defines a distribution on $\Omega = (0, \infty)$. Find its degree.

Given $T \in \mathcal{D}'(\Omega)$ we define its derivative $D^\alpha T \in \mathcal{D}'(\Omega)$ with

$$D^\alpha T(\varphi) = (-1)^{|\alpha|}\, T(D^\alpha \varphi).$$

- B1. Prove that $D^\alpha T$ is a distribution.
- B2. Prove that if $u \in C_c^\infty(\Omega)$ and $T_u(\varphi) = \int_\Omega u(x)\, \varphi(x)\, \mathrm{d}x$ then $D^\alpha T_u(\varphi) = T_{D^\alpha u}(\varphi)$ (that is: distributional derivative coincides with the classical one!).
- B3. Compute distributional derivative of the function $|x|$ on $(-1, 1)$.
- B4. Compute distributional derivative of the function $\mathbb{1}_{x>0}$ on $(-1, 1)$.
- B5. Let Φ be a fundamental solution to Laplace equation. Prove that $-\Delta\Phi = \delta_0$ in the sense of distributions. *Hint:* recall the formula proved in the lecture for all $u \in C^2(\Omega)$:

$$u(x) = -\int_{\partial\Omega} u(y)\, \frac{\partial \Phi}{\partial \mathbf{n}}(y-x)\; \mathrm{d}S(y) + \int_{\partial\Omega} \Phi(y-x)\, \frac{\partial u}{\partial \mathbf{n}}(y)\; \mathrm{d}S(y) - \int_\Omega \Phi(y-x)\, \Delta u(y)\; \mathrm{d}y.$$

- B6. Prove that distributional derivatives satisfy Schwarz lemma (their order can be interchanged).

A Sobolev space $W^{k,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ such that for $|\alpha| \le k$ the distributional derivative $D^\alpha u \in L^p(\Omega)$. It turns out to be a Banach space with an obvious norm.

- C1. Write explicitly integral identity satisfied by derivatives of Sobolev functions.
- C2. For which $1 \le p \le \infty$, $|x| \in W^{1,p}(-1, 1)$?
- C3. For which $1 \le p \le \infty$, $\mathbb{1}_{x>0} \in W^{1,p}(-1, 1)$?