Introduction to PDEs (SS 20/21), Problem Set B2

Sobolev spaces: examples, calculus and smooth approximation

Compiled on 12/05/2021 at 11:58pm

Let Ω be an open doman in $\mathbb{R}^d$. We say that $u \in L^1_{loc}(\Omega)$ is weakly differentiable if its distributional derivatives belongs to $L^1_{loc}(\Omega)$. The space of weakly differentiable functions $u \in L^p(\Omega)$ with weak derivative $Du \in L^p(\Omega)$ forms a Banach space called Sobolev space. It is known that weak derivatives are unique (provided they exist) and they satisfy usual calculus rules (sum, product, multiplication with scalars) assuming that all the terms under integrals are well defined.

Examples of Sobolev functions

A1. Prove that $|x| \in W^{1,p}(-1,1)$ for all $1 \le p \le \infty$.

A2. Prove that $\mathbb{1}_{x>0} \notin W^{1,p}(-1,1)$ for all $1 \le p \le \infty$.

A3. Consider $u(x) = \mathbb{1}_{|x|<1}$ in $\mathbb{R}^d$.

- (A) Prove that $u \in W^{1,p}(B_1(0))$ where $B_1(0) = \{|x| < 1\}$ and find its weak gradient.
- (B) Prove that $u \notin W^{1,p}(\mathbb{R}^d)$.
- (C) Find distributional gradient of u in $\mathbb{R}^d$, i.e. find formula for the distribution $(\partial_{x_i} T_u)(\phi)$ where $T_u(\phi) = \int_{\mathbb{R}^d} u(x)\,\phi(x)\mathrm{d}x$. Is there a function $v_i \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$(\partial_{x_i} T_u)(\phi) = \int_{\mathbb{R}^d} v_i(x)\,\phi(x)\mathrm{d}x?$$

A4. Prove that if $u \in C^k(\overline{\Omega})$ and Ω is bounded then $u \in W^{k,p}(\Omega)$ for all $1 \le p \le \infty$.

A5. Find all α, d and p such that $u(x) = |x|^{-\alpha}$ belongs to $W^{1,p}(B_1(0))$ with $B_1(0) \subset \mathbb{R}^d$.

A6. Let $\{r_k\}_{k\in\mathbb{N}}$ be a dense, countable subset of $B_1(0)$. Prove that for α, p and d as in Problem A4 function

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k}\,|x - r_k|^{-\alpha}$$

belongs to $W^{1,p}(B_1(0))$ yet it is unbounded in every open subset of $B_1(0)$.

A7. Let U be an open square in $\mathbb{R}^2$, i.e. $U = \{(x_1, x_2) : |x_1| < 1, |x_2| < 1\}$. Consider function

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, |x_2| < x_1, \\ 1 + x_1 & \text{if } x_1 < 0, |x_2| < -x_1, \\ 1 - x_2 & \text{if } x_2 > 0, |x_1| < x_2, \\ 1 + x_2 & \text{if } x_2 < 0, |x_1| < -x_2. \end{cases}$$

Find all $1 \le p \le \infty$ so that $u \in W^{1,p}(U)$.

More calculus rules

B1. If $U \subset \Omega$ and $u \in W^{k,p}(\Omega)$ then $u \in W^{k,p}(U)$ with the same derivative.

B2. (chain rule) Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1 Lipschitz function, $1 \le p \le \infty$ and Ω be a bounded domain. If $u \in W^{1,p}(\Omega)$ then $F(u) \in W^{1,p}(\Omega)$ with $\partial_{x_i} F(u) = F'(u)\, u_{x_i}$.

B3. Let $1 \le p < \infty$ and Ω be a bounded domain. Prove that if $u \in W^{1,p}(\Omega)$ then its modulus $|u|$, positive part $u^+ := u\,\mathbb{1}_{u>0}$ and negative part $u^- := -u\,\mathbb{1}_{u<0}$ belong to $W^{1,p}(\Omega)$. Find their derivatives in terms of Du.

Hint: For $\varepsilon > 0$ consider function $F_\varepsilon(z) = (z^2+\varepsilon^2)^{1/2} - \varepsilon$. Prove that $F_\varepsilon(z)$ is Lipschitz with constant 1 and C^1 with $|F'_\varepsilon| \le 1$. Moreover $F_\varepsilon(z) \to |z|$ and $F'_\varepsilon(z) \to \mathrm{sgn}(z)\,\mathbb{1}_{z\neq 0}$ as $\varepsilon \to 0$. Similarly, consider function $G_\varepsilon(z) = \begin{cases} (z^2+\varepsilon^2)^{1/2} - \varepsilon & \text{if } z \ge 0, \\ 0 & \text{if } z < 0. \end{cases}$

Approximation of Sobolev functions and its consequences

C1. (Poincare) Let $u \in W_0^{1,p}(\Omega)$ where Ω is a bounded domain. Prove that

$$\|u\|_p \le C(\Omega)\, \|Du\|_p.$$

Conclude that there is an equivalent norm on $W_0^{1,p}(\Omega)$, namely $u \mapsto \|Du\|_p$.

C2. Prove that Poincare inequality is satisfied for unbounded Ω if it is directionally bounded i.e. there is i such that $\sup_{x\in\Omega} |x_i| < \infty$.

C3. Prove that Poincare inequality is not true for $u \in W^{1,p}(\Omega)$.

C4. Prove that on $H_0^2(\Omega)$, the map $u \mapsto \|\Delta u\|_2$ defines an equivalent norm, i.e. there is a constant C (independent of u!) such that

$$\|\Delta u\|_2 \le \|u\|_{H^2}, \qquad \|u\|_{H^2} \le C\, \|\Delta u\|_2.$$

Hint: use both Poincare inequality and smooth approximation.

Hence, weak Laplacian contains all the information about second order weak derivatives of u. This was not the case for strong derivatives, cf. Schauder estimates!

C5. Let $1 < p < \infty$ and $u \in W^{1,p}(0,1)$. The target of this exercise is to prove that then u has a continuous version on $[0,1]$ and

$$|u(x) - u(y)| \le |x-y|^{1-1/p} \|u'\|_p, \qquad\qquad u(x) = u(y) + \int_y^x u'(z)\, \mathrm{d}z.$$

This will be done in the series of small steps outlining very important (!) argument for finding a continuous representative.

(A) Prove that if $u \in C^1[0,1]$ then

$$|u(x) - u(y)| \le |x-y|^{1-1/p} \|u'\|_p.$$

(B) Prove that if $u \in W^{1,p}(0,1)$ then for a.e. $x, y \in (0,1)$ we have

$$|u(x) - u(y)| \le |x-y|^{1-1/p} \|u'\|_p.$$

(C) Fix $\delta > 0$ and consider $u^\varepsilon = u * \eta_\varepsilon$ with $\varepsilon < \delta/2$. Prove that for $x \in [\delta, 1-\delta]$, the sequence $\{u^\varepsilon(x)\}$ satisfies assumptions of Arzela-Ascoli theorem.

(D) Using properties of mollifiers conclude that u has a continuous representative for $(0,1)$ and for $x, y \in (0,1)$ we have

$$|u(x) - u(y)| \le |x-y|^{1-1/p} \|u'\|_p, \qquad u(x) = u(y) + \int_y^x u'(z) \, \mathrm{d}z.$$

(E) Extend the representative (in a continuous way) as well as results of (D) to $[0,1]$.

C6. Prove interpolation inequality

$$\int_\Omega |Du|^2 \, \mathrm{d}x \le C \left(\int_\Omega |D^2 u|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |u|^2 \, \mathrm{d}x \right)^{1/2},$$

first for $u \in C_c^\infty(\Omega)$ and then for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.