Introduction to PDEs (SS 20/21), Problem Set B3

Sobolev spaces: important results

Compiled on 21/05/2021 at 3:49pm

This list contains exercises which should help to understand (formulation, special cases, proofs, applications) of important results about Sobolev spaces: smooth approximation, extension theorem, Sobolev embeddings and Reillich-Kondrachov theorem.

Smooth approximation

Theorem: Let Ω be bounded, $\partial\Omega$ be C^1 and $1 \leq p < \infty$. Then, for all $u \in W^{k,p}(\Omega)$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

A1. Let $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$. Prove that if η_ε is a usual mollification kernel,

$$\partial_{x_i}(u * \eta_\varepsilon) = (\partial_{x_i} u) * \eta_\varepsilon = (\partial_{x_i}\eta_\varepsilon) * u$$

where $(\partial_{x_i} u)$ denotes weak derivative of u!

A2. Let $u \in W^{1,p}(\Omega)$ and suppose that $Du = 0$ a.e. in Ω. Prove that u is constant.

A3. Let $u \in W^{1,p}_0(\Omega)$. Prove that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty_c(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Compare with the case $u \in W^{1,p}(\Omega)$.

Extension theorem

Theorem: If $1 \leq p \leq \infty$, Ω is bounded and $\partial\Omega$ is C^1. Choose V such that U is compactly supported in V. Then, there exists a bounded linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that $Eu = u$ a.e. in Ω and Eu has support in V.

B1. Let $u = \mathbb{1}_{[0,1]} \in W^{1,1}(0,1)$. Extend u to $W^{1,1}(\mathbb{R})$.

B2. Let $u \in W^{1,p}_0(\Omega)$. Prove that the trivial extension $\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases}$ belongs to $W^{1,p}(\mathbb{R}^d)$ so that in this case we don't need extension theorem.

B3. Discuss extension results for $C^k(\overline{\Omega})$ cf. Whitney Extension Theorem.

Trace operator

Theorem: If $1 \leq p < \infty$, Ω is bounded and $\partial\Omega$ is C^1 there exists a bounded linear operator

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Moreover, $u \in W^{1,p}_0(\Omega)$ if and only if $Tu = 0$.

C1. Prove that $1 \notin W^{1,p}_0(\Omega)$ so that the inclusion $W^{1,p}_0(\Omega) \subset W^{1,p}(\Omega)$ is strict.

C2. Prove that there is no trace operator on $L^p(\Omega)$: prove that there does not exists a bounded linear operator

$$T : L^p(\Omega) \to L^p(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega}) \cap L^p(\Omega)$.

C3. (trace in 1D and $1 < p < \infty$) Prove that the functional $\varphi : W^{1,p}(0,1) \to \mathbb{R}$ defined with $\varphi(u) = u(0)$ is continuous. *Hint:* use continuous version of u.

C4. This problem shows that all integration-by-parts formulas hold true for Sobolev functions if their boundary value is replaced with its trace. For example, prove that

$$\int_\Omega D_j u(x)\, v(x) + \int_\Omega u(x)\, D_j v(x) = \int_{\partial\Omega} (Tu)(x)\,(Tv)(x)\, n_j(x)$$

for $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$, where Tu and Tv denotes traces of u and v, $1 < p < \infty$ and p' is the usual Holder conjugate.

Sobolev embeddings

Theorem (Sobolev): If $1 \le p < n$, Ω is bounded and $\partial\Omega$ is C^1 then $W^{1,p}(\Omega)$ is continuously embedded in L^q where $q < p^*$.

Theorem (Morrey): If $p > n$, Ω is bounded and $\partial\Omega$ is C^1 then $W^{1,p}(\Omega)$ is continuously embedded in $C^{0,\gamma}$ for some $\gamma \in (0,1)$.

Reillich-Kondrachov compactness

Theorem (R-K): If $1 \le p < n$, Ω is bounded and $\partial\Omega$ is C^1 then $W^{1,p}(\Omega)$ is compactly embedded in L^q where $q < p^*$.

E1. Prove R-K theorem for $p = 1$ and $n = 1$, i.e. $W^{1,1}(I)$ is compactly embedded in $L^1(I)$. Follow the steps:

(A) Start with a sequence $\{u_n\}_{n\in\mathbb{N}}$ bounded in $W^{1,1}(I)$. Fix a bounded interval J and extend u_n to $W^{1,1}(\mathbb{R})$ such that support of u_n lies in J.

(B) Consider $u_n^\varepsilon = u_n * \eta_\varepsilon$. Prove that $u_n^\varepsilon \to u_n$ in $L^1(J)$, uniformly in n.

(C) Prove that if $\varepsilon > 0$ is fixed, the sequence $\{u_n^\varepsilon\}_{n\in\mathbb{N}}$ satisfies assumptions of Arzela-Ascoli Theorem.

(D) Fix $\delta > 0$. Prove that there exists a subsequence $\{u_{n_k}\}$ such that

$$\limsup_{n_k, n_l \to \infty} \|u_{n_k} - u_{n_l}\|_{L^1(J)} \le \delta.$$

(E) Conclude using diagonal argument and completeness of $L^1(J)$.

This is the case not commented in the book of Evans.

E2. Go through the proof in Problem E1 and explain where one needs to use Sobolev embeddings in the general case.

E3. Prove the following useful version of R-K theorem by considering $p < n$ and $p \ge n$: if Ω is a bounded domain with C^1 boundary, $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$.

E4. Prove that $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, no matter whether boundary of Ω is smooth or not.

E5. Formulate this in terms of compact operators from functional analysis.

E6. Compare R-K theorem with Arzela-Ascoli theorem.

E7. By a usual contradiction argument prove Poincare inequality with averages:

$$\|u-(u)_\Omega\|_{L^p(\Omega)} \leq C(\Omega)\,\|Du\|_{L^p(\Omega)}.$$

E8. Let $u \in W^{1,p}(\Omega)$ where Ω is bounded and connected domain. Prove that if u vanishes on $U \subset \Omega$ and $|U| > 0$ then

$$\|u\|_{L^p(\Omega)} \leq C(\Omega, U)\,\|Du\|_{L^p(\Omega)}.$$

E9. Deduce usual Poincare inequality for $u \in W_0^{1,p}(\Omega)$:

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\,\|Du\|_{L^p(\Omega)}.$$

E10. Prove explicit form of Poincare inequality for balls: if $u \in W^{1,p}(B(x,r))$

$$\|u-(u)_{B(x,r)}\|_{L^p(B(x,r))} \leq C\,r\,\|Du\|_{L^p(B(x,r))}$$

and the constant C is independent of r. *Hint:* Consider $v(y) = u(x + r\,y)$ and prove that $v \in W^{1,p}(B(0,1))$.

E11. Prove that if $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ then u belongs to the space of functions of bounded mean oscillation (BMO), i.e.

$$|u|_{BMO} := \sup_{B(x,r)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u-(u)_{B(x,r)}| < \infty$$

This is Sobolev embedding for $p = n$.