Introduction to PDEs (SS 20/21), Problem Set C1

Introduction to weak formulations

Compiled on 19/05/2021 at 6:46pm

This is a list of topics for discussion rather than list of problems but I believe it is somehow necessary in the first course on PDEs.

1. Consider the following three formulations of Poisson equation:

   (A) $u \in C^2(\Omega)$, $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$,

   (B) $u \in C^1(\Omega)$, $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \varphi\, f$ for all $\varphi \in C_c^\infty(\Omega)$ and $u = 0$ on $\partial\Omega$,

   (C) $u \in C(\Omega)$, $\int_\Omega u\, \Delta \varphi = - \int_\Omega \varphi\, f$ for all $\varphi \in C_c^\infty(\Omega)$ and $u = 0$ on $\partial\Omega$

   Prove that if u solves (A) then it solves (B) and if u solves (B) then it solves (C).

2. Speaking about formulations (A) - (C), for which of them you expect existence is easier to prove? Similarly, for which of them you expect uniqueness is easier to prove?

3. Consider formulations of Poisson equations in Sobolev spaces:

   (A*) $u \in H^2(\Omega) \cap H^1_0(\Omega)$, $-\Delta u = f$ a.e.,

   (B*) $u \in H^1_0(\Omega)$, $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\, \varphi$ for all $\varphi \in C_c^\infty(\Omega)$.

   Prove that if u solves (A) then u solves (A*). Similarly, if u solves (B) then u solves (B*).

4. Prove that (B*) is equivalent with

   (B**) $u \in H^1_0(\Omega)$, $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\, \varphi$ for all $\varphi \in H^1_0(\Omega)$.

   This will be very useful for formulating PDEs in Hilbert spaces.

5. Sometimes higher regularity may upgrade weak formulation to the stronger one. For instance, suppose that u solves (B*) and $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Prove that u solves (A*).

6. Let $b, c \in L^\infty(\Omega)$. For the following PDE

   $$-\Delta u + c(x)\, u + b(x) \cdot \nabla u = 0 \text{ in } \Omega, \qquad u = 0 \text{ in } \partial\Omega,$$

   find weak formulation that makes sense for $u \in H^1_0(\Omega)$.

Conclusion: A good weak formulation:

(A) agrees with a strong (classical) formulation,

(B) has a solution (existence) and it is easy to prove so,

(C) has the unique solution (uniqueness) and it is easy to prove so.

As we will see, for elliptic equations formulation (B*) has all these features.