Introduction to PDEs (SS 20/21), Problem Set C2

Theory for elliptic equations in Hilbert spaces

Compiled on 08/06/2021 at 12:40 Noon

Lax-Milgram Lemma Let H be a Hilbert space. Let $a : H \times H \to \mathbb{R}$ be a continuous bilinear which is coercive, i.e.

$$a(u,u) \geq c\,\|u\|^2 \text{ for some constant } c.$$

Then, for each $l \in H^*$ there is exactly one $u \in H$ such that $a(u,v) = l(v)$ for all $v \in H$.

We proved this in Functional Analysis class under additional assumption that $a(u,v) = a(v,u)$. Then a defines a scalar product on H and the conclusion follows from Riesz Representation Theorem.

1. We first use Lax-Milgram lemma to establish theory for Poisson equation $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. We say that $u \in H^1_0$ is a weak solution if for all $\varphi \in H^1_0(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\,\varphi.$$

   (A) Prove that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution then u is a weak solution.
   (B) Prove that $H^1(\Omega)$ and $H^1_0(\Omega)$ are Hilbert spaces.
   (C) Prove that $a(u,\phi) := \int_\Omega \nabla u \cdot \nabla \varphi$ is a continuous bilinear coercive form on $H = H^1_0(\Omega)$.
   (D) Prove that $\varphi \mapsto \int_\Omega f\,\varphi$ is a continuous functional on $H = H^1_0(\Omega)$.
   (E) Use Lax-Milgram lemma to establish existence and uniquess for Poisson equation.
   (F) Btw, prove that it is sufficient to consider test functions $\varphi \in C^\infty_c(\Omega)$ in the definition of weak solution.

2. Let $\lambda \in \mathbb{R}$, $f \in L^2(\Omega)$ and consider equation

$$-\Delta u + \lambda\, u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

   (A) Define weak solutions in $H^1_0(\Omega)$
   (B) For which λ one can use Lax-Milgram to obtain well-posedness of this problem in $H^1_0(\Omega)$?
   (C*) One can prove that in fact the unique solution to (1) belongs to $H^2(\Omega) \cap H^1_0(\Omega)$. Consider unbounded operator $A = \Delta$ with $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Prove that $(0,\infty)$ belongs to the resolvet of A. Moreover, if $R_\lambda : L^2(\Omega) \to L^2(\Omega)$ is resolvent operator we have an estimate

$$\|R_\lambda\| \leq \frac{C}{\lambda} \qquad (\lambda > 0),$$

   the constant C is independent of λ. This shows that A satisfies assumptions of Hille-Yosida theorem.

3. Let $f \in L^2(\Omega)$. We say that $u \in H^2_0(\Omega)$ is a weak solution to the biharmonic equation

$$\Delta^2 u = f \text{ in } \Omega, \qquad u = \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega$$

   provided for all $\varphi \in H^2_0(\Omega)$ we have

$$\int_\Omega \Delta u\, \Delta \varphi = \int_\Omega f\,\varphi.$$

(A) Suppose that $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution to the biharmonic equation. Prove that it is also a weak solution. *Hint:* If $u \in H_0^2(\Omega)$ then $u_{x_i} \in H_0^1(\Omega)$.

(B) Prove that there exists the unique weak solution of biharmonic equation.

4. One can also consider problems with Neumann boundary condition, i.e.

$$-\Delta u = f \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega.$$

We say that $u \in H^1(\Omega)$ is a weak solution to this problem if for all $\varphi \in H^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\, \varphi.$$

(A) Prove that if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a strong solution then it is also a weak solution.

(B) Prove that if there exists a solution u, we have $\int_\Omega f = 0$.

(C) Consider $\mathcal{H} = \{u \in H^1(\Omega) : \int_\Omega u = 0\}$ and prove it is a closed subspace of $H^1(\Omega)$. Prove that there exists the unique weak solution in $\mathcal{H}$.

(D) Explain why without restricting to $\mathcal{H}$ one cannot expect uniqueness.

Hint: Recall Poincare inequality with averages.

5. Prove stability of solutions to Poisson equations in $H_0^1(\Omega)$, i.e. if u_1, u_2 are weak solutions to

$$-\Delta u_i = f_i \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega$$

then

$$\|u_1 - u_2\|_{H^1} \leq C(\Omega)\, \|f_1 - f_2\|_2.$$

Hint: Recall ε-Cauchy-Schwartz inequality.

6. In what follows, $H^{-1}(\Omega) = (H_0^1(\Omega))^*$. Prove that:

(A) we have a continuous embedding $L^2 \subset H^{-1}(\Omega)$,

(B) in fact, a stronger result holds true: $L^q(\Omega) \subset H^{-1}(\Omega)$ with q...,

(C) prove that if $f \in L^2(\Omega)$ then $\partial_{x_i} f$ (in the sense of distributions!) belongs to $H^{-1}(\Omega)$,

(D) conclusion of Lax-Milgram lemma applied to Poisson equation is still valid if $f \in L^2(\Omega)$ is replaced with $f \in H^{-1}$.