Solution to the 1st Special Problem

Assume contrary, i.e. there exists a countable infinite family $\{f_i\}$ consisting of elements from Hamel basis of E, s.t. the projection functionals P_i are continuous. As P_i are nontrivial, then $||P_i|| \neq 0$, so we may define

$$Q_i := \frac{i}{||P_i||} \cdot P_i.$$

The functionals Q_i are continuous with $||Q_i|| = i$. (*)

Note that for each $x \in E$ we have $\sup_{i\in\mathbb{N}} |Q_i(x)| < \infty$, as $Q_i(x)$ is nonzero only for a finite number of i's (because x expressed as a sum of elements from Hamel basis has only finitely many nonzero components). As E is a Banach space, then by Uniform Boundedness Principle follows that $\sup_{i\in\mathbb{N}} ||Q_i|| < \infty$, which is a contradiction with (*).

Solution to the 2nd Special Problem

For a fixed $u \in H$ by Riesz theorem the functional $a(u, \cdot)\colon H \to \mathbb{R}$ is of the form $\langle u', \cdot\rangle$ for uniquely determined u'. Therefore, we may define $L\colon H \to H$ as $L(u) := u'$.

Note that L is linear, bounded, injective and surjective:

- If $\beta \in \mathbb{R}$, then for a fixed $u \in H$ and any $x \in H$ holds
$$\langle L(\beta u), x\rangle = a(\beta u, x) = \beta a(u, x) = \beta\langle L(u), x\rangle = \langle \beta L(u), x\rangle,$$
so $L(\beta u) = \beta L(u)$. Moreover, if u, $v \in H$, then for any $x \in H$
$$\langle L(u + v), x\rangle = a(u + v, x) = a(u, x) + a(v, x) = \langle L(u), x\rangle + \langle L(v), x\rangle = \langle L(u) + L(v), x\rangle,$$
so $L(u + v) = L(u) + L(v)$, so L is indeed linear. (Summing up: linearity of L follows from a being linear in first variable).

- Since a is continuous, then there is $D > 0$ s.t. for any u, $v \in H$ holds $a(u, v) \leqslant D \cdot ||u|| \cdot ||v||$. Therefore
$$||L(u)||^2 = \langle L(u), L(u)\rangle = a(u, L(u)) \leqslant D \cdot ||u|| \cdot ||L(u)|| \implies ||L(u)|| \leqslant D||u||,$$
so L is bounded. (Here we used just the continuity of a).

- If L wasn't injective, then for some $u \neq 0$ we would have $L(u) = 0$. Then
$$0 < C \cdot ||u||^2 \leqslant a(u, u) = \langle L(u), u\rangle = \langle 0, u\rangle = 0,$$
a contadiction. (Here we used the coercivity of a).

- Firstly, let's prove that $\operatorname{Im} L$ is closed. Let $v, u_i \in H$ be such that $L(u_i) \xrightarrow{i\to\infty} v$. Note that for any $u \in H$ holds
$$C \cdot ||u||^2 \leqslant a(u, u) = \langle L(u), u\rangle \leqslant ||L(u)|| \cdot ||u|| \implies ||u|| \leqslant \frac{1}{C} \cdot ||L(u)||,$$
so if $L(u_i)$ has a limit, then u_i has a limit as well (denote it by u), but since L is continuous, then $v = L(u)$ is a limit of $L(u_i)$. Therefore $v \in \operatorname{Im} L$, as desired.

  If L was not surjective, then $\operatorname{Im} L$ is a proper closed subspace of H. This means that there exists a nonzero $m \in H$, s.t. $m \in (\operatorname{Im} L)^\perp$. Then $0 < C \cdot ||m||^2 = a(m, m) = \langle L(m), m\rangle = 0$, as $m \perp L(m)$. This is a contradiction, so L is indeed surjective.

Let $l \in H^*$. By Riesz representation theorem there is a unique $v_l \in H$, such that
$$\langle v_l, \cdot\rangle = l(\cdot).$$
Then, by our reasoning, there is a unique $u_l \in H$ (namely $u_l = L^{-1}(v)$), such that $\langle v_l, \cdot\rangle = a(u_l, \cdot)$. Altogether, we conclude that there is a unique $u := u_l$ such that $a(u, \cdot) = l(\cdot)$, as desired.

Solution to the 4th Special Problem

1. Let $\varphi \in X^*$. By problem H4 from PS6 we know there exists $f \in X^{**}$ such that

$$||f|| = 1 \quad \text{and} \quad ||\varphi|| = f(\varphi).$$

Let $i\colon X \to X^{**}$ be the canonical isometry between X and X^{**}. Let $x_0 = i^{-1}(f) \in X$. It is clear that $||x_0|| = 1$ and

$$f(\varphi) = i(x_0)(\varphi) \overset{\text{def. of } i}{=} \varphi(x_0),$$

so indeed $||\varphi|| = \varphi(x_0)$, as we wanted.

2. Let M be a closed (strictly contained) subspace of X. By H10 from PS6 we get that there is $\varphi \in X^*$ such that $\varphi \neq 0$, $||\varphi|| = 1$ and $\varphi(x) = 0$ for all $x \in M$. By point 1. we conclude there is $x_0 \in X$ satisfying

$$||x_0|| = 1 \quad \text{and} \quad \varphi(x_0) = ||\varphi|| = 1.$$

Let $m \in M$. Then

$$1 = \varphi(x_0) = \varphi(x_0) - \varphi(m) \leqslant ||\varphi|| \cdot ||x_0 - m|| = ||x_0 - m||,$$

so $||x_0 - m|| \geqslant 1$ for all $m \in M$, which implies $\operatorname{dist}(x_0, M) = 1$.

3. Let $u \in X$. We claim that $\operatorname{dist}(u, M) = |\int_0^1 u|$. Let $c = \int_0^1 u$ and $m \in M$. We see that

$$||u - m|| = \int_0^1 ||u - m||\mathrm{d}t \geqslant \int_0^1 |(u - m)(t)|\mathrm{d}t \geqslant \left|\int_0^1 (u - m)(t)\mathrm{d}t\right| = \left|\int_0^1 u(t)\mathrm{d}t\right| = |c|,$$

so $\operatorname{dist}(u, M) \geqslant |c|$. Now, let $f_n\colon [0,1] \to \mathbb{R}$ for $n = 1, 2, \ldots$ be defined as follows

$$f_n(x) = \begin{cases} cx \cdot \frac{2n^2}{2n-1}, & \text{if } x \in [0, \frac{1}{n}], \\ c \cdot \frac{2n}{2n-1}, & \text{otherwise.} \end{cases}$$

Note that $\int_0^1 f_n = c$, so $u - f_n \in M$. Then

$$||u - (u - f_n)|| = ||f_n|| = |c| \cdot \frac{2n}{2n-1} \xrightarrow{n \to \infty} |c|,$$

implying $\operatorname{dist}(u, M) \leqslant |c|$, so indeed $\operatorname{dist}(u, M) = |c|$.

4. The set M is a closed (strictly contained) linear subspace of X. Let $u \in X$ satisfy $||u|| = 1$. Since u is continuous and $u(0) = 0$, then

$$\exists \epsilon > 0 \ \forall x \in (0, \epsilon) \ \ |u(x)| < \frac{1}{2}.$$

This together with $||u|| = 1$ yields $|\int_0^1 u| < 1$, which by point 3. gives $\operatorname{dist}(u, M) < 1$. In other words, Riesz Lemma does not hold in X with the supremum norm.