Functional Analysis (WS 19/20)

(Special Problems)

Rules: Each problem has assigned deadline for submission of the solution. If the problem remains unsolved, the deadline is extended and some hints are provided. Each problem is worth 2 points in the tutorial classification (added independently of regular homeworks, active class participation).

Contents

1 Projection functionals onto Hamel basis elements are usually not continuous.

Announced: 4/11/2019, Deadline: 21/11/2019.

Let $(E, \|\cdot\|)$ be an infinite-dimensional normed space. Recall the following facts:

- if $(E, \|\cdot\|)$ is a Banach space then using Baire Category Theorem we proved that Hamel basis of E is not countable (Problem B1, Problem Set 3),
- projections onto vectors from Hamel basis are not continuous in general – we considered space of polynomials with L^1 norm and projection of $(x-1)^n$ onto vector 1 (Problem A2, Problem Set 2).

In what follows, we will see that not so many such functionals can be actually continuous. Let $(E, \|\cdot\|)$ be a Banach space and $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ be its Hamel basis. Choose countable subset $\{f_i\}_{i\in\mathbb{N}} \subset \{e_\alpha\}_{\alpha\in\mathcal{A}}$ and consider projection functionals $P_i : E \to \mathbb{R}$ onto f_i such that

$$P_i(x) = a \text{ whenever Hamel decomposition of } x \text{ is } x = af_i + ...$$

Prove that at most finitely many of P_i can be continuous.

2 Lax–Milgram Lemma for nonsymmetric bilinear forms.

Announced: 9/11/2019, Deadline: 5/12/2019.

Let $(H, \|\cdot\|)$ be a Hilbert space. Suppose that $a : H \times H \to \mathbb{R}$ is a bilinear continuous form that is coercive, i.e. there is a constant $C > 0$ such that $a(u, u) \geq C\|u\|_H^2$. Moreover, let $l \in H^*$. Prove that there exists uniquely determined $u \in H$ such that

$$a(u, v) = l(v)$$

holds for all $v \in H$.

Note: We have already seen this result in case of symmetric bilinear forms (Problem R3, Problem Set 5). *Hint:* Define $A : H \to H^*$ and try to use Riesz Representation Theorem.

3 Explicit construction of Brownian motion (counts as 2 special problems).

Announced: 9/11/2019, Deadline: 5/12/2019.

The following problem is supposed to present some applications of Hilbert spaces theory in Probability. Much of the current research effort is put on the topics around Brownian motion, i.e. family of random variables $\{B_t\}_{t\in[0,1]}$ indexed with time $t \in [0, 1]$. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that the family of random variables $\{B_t\}_{t\in[0,1]}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ if

1. $B_0 = 0$ almost surely,

2. $B_t - B_s$ is distributed normally with $\mathcal{N}(0, t-s)$ for $t > s$ (we say it has stationary increments),

3. B_t has independent increments, i.e. for all $0 \le t_1 \le t_2 \le ... \le t_n \le 1$ we have that random variables $B_{t_n} - B_{t_{n-1}}$, $B_{t_{n-1}} - B_{t_{n-2}}$, ..., B_{t_1} are independent,

4. B_t has almost surely continuous paths, i.e. for a.e. $\omega \in \Omega$, the map $[0,1] \ni t \mapsto B_t(\omega)$ is continuous.

One can read a lot about motivations for considering such process in countless Internet materials. Standard construction of Brownian motion is based on the Kolmogorov Existence Theorem which is extremely technical and measure-theoretical so it does not say too much about nature of the constructed process. In what follows, we present *explicit* construction of Brownian motion due to Ciesielski. This requires basic notions of Probability Theory like characteristic functions, convergence a.e., convergence in probability, etc.

1. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(0,1)$. We set $\psi_n(t) = \int_0^t \phi_n(s)\,ds$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be i.i.d. random variable with $\mathcal{N}(0,1)$ distribution on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that the series
$$B_t = \sum_{n=1}^{\infty} \xi_n \psi_n(t)$$
converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. This is our candidate for Brownian motion. *Remark:* Using Levy Theorem (series of independent random variables converges almost surely if and only if it converges in probability), one can deduce that B_t converges also almost surely.

2. Check that B_t has independent and stationary increments. *Hint:* To check independence of increments, it may be helpful to study characteristic functions.

3. We need to prove continuity of paths. To this end, we use Haar functions. We set $\phi_0 = 1$. Moreover, for all $n \in \mathbb{N}$ and $1 \le k \le 2^n$ we define
$$\phi_{n,k}(t) = \begin{cases} +2^{n/2} & \text{if } \frac{k-1}{2^n} \le t \le \frac{k-(1/2)}{2^n}, \\ -2^{n/2} & \text{if } \frac{k-(1/2)}{2^n} < t \le \frac{k}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$
Check that the system $\{\phi_{n,k}\}_{n\in\mathbb{N}, 1\le k\le 2^n} \cup \{\phi_0\}$ is an orthonormal basis of $L^2(0,1)$.

4. Prove that the series in point (1) with Haar functions used as an orthonormal basis of $L^2(0,1)$ converges uniformly on $[0,1]$ almost surely. Hence, the limit is almost surely continuous. *Hint:* Use Borel-Cantelli Lemma and estimate Gaussian tails.

4 Special cases of Riesz Lemma.

Announced: 19/11/2019, Deadline: 19/12/2019.

We have seen the following result in the tutorials. Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ a closed linear subspace that is strictly contained. For each $\alpha \in (0,1)$ there is x_α such that $\|x_\alpha\| = 1$ and $\operatorname{dist}(x_\alpha, M) \ge \alpha$. Here, we will see that for reflexive Banach spaces, case $\alpha = 1$ is also true. Moreover, we prove that otherwise the result with $\alpha = 1$ conclusion may fail. This is accompanied with few results that are of independent interest.

1. Let $(X, \|\cdot\|)$ be a reflexive Banach space (i.e. $X^{**} = X$ up to the canonical isometric isomorphism[1]). Prove that the norm of bounded linear functionals from X^* is attained, i.e. for any $\varphi \in X^*$ there is some $x \in X$ with $\|x\| = 1$ such that $\varphi(x) = \|\varphi\|$.
2. Let $(X, \|\cdot\|)$ be a reflexive Banach space. Prove that Riesz Lemma holds for $\alpha = 1$.
3. Consider $X = \{f \in C[0,1] : f(0) = 0\}$ with the supremum norm and let $\varphi(f) = \int_0^1 f(s)\, ds \in X^*$. Let $M = \ker \varphi$. For given $u \in X$, compute $\mathrm{dist}(u, M)$. *Remark:* It is quite remarkable that the result of this computation can be easily generalized to other functionals.
4. Prove that Riesz Lemma does not hold in X with the supremum norm.

5 Only linear functions are weakly continuous.

Announced: 19/11/2019, Deadline: 19/12/2019.

In nonlinear analysis, it is important to be able to extract converging (in some, usually, weak sense) subsequences from a bounded ones by means of some Banach–Alaoglu type theorems. However, in applications, one usually deals with functions evaluated on this sequences. For instance, when looking for the minimum of a continuous function $f : [0,1] \to \mathbb{R}$, we extract converging subsequence $x_{n_k} \subset [0,1]$ but then, we are actually interested in the properties of $f(x_{n_k})$.

In the following, we prove that if $a : \mathbb{R} \to \mathbb{R}$ is weakly continuous, then a is an affine functions. Hence, in general weak continuity cannot be expected from nonlinear functions.

1. Let $u : \mathbb{R} \to \mathbb{R}$ be a bounded, 1-periodic function. Let $u_n(x) = u(nx)$. Prove that
$$u_n \rightharpoonup m = \int_0^1 u(x)\, dx \text{ in } L^2(A)$$
for every open, bounded set $A \subset \mathbb{R}$.
2. Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function such that
$$a(f_n) \rightharpoonup a(f) \text{ in } L^2(0,1) \text{ whenever } f_n \rightharpoonup f \text{ in } L^2(0,1).$$
Prove that a is affine, i.e. there are α and β such that $a(z) = \alpha z + \beta$.

6 Banach-Alaoglu Theorem for reflexive and separable spaces (with applications).

Announced: 19/11/2019, Deadline: 19/12/2019.

In the following, we will see that although nonlinearities do not preserve weak limits (as was discussed in Problem 5), weak convergence can be still used to study minimization problems for the nonlinear functions defined on infinite dimensional Banach spaces.

[1] In the original version of the problem it was stated that $X^{**} = X$ up to an isometric isomorphism. This is not true. Due to James (1951), there are Banach spaces X isometrically isomorphic to X^{**} but not reflexive – see James' space on Wikipedia. Reflexivity is based on an isometric isomorphism that is precisely a point evaluation. To be clear, we say that X is reflexive if there exists isometric isomorphism $J : X \to X^{**}$ such that for all $x \in X$ and $\varphi \in X^*$ we have $(Jx)(\varphi) = \varphi(x)$. That was pointed out by Kuba Woźnicki.

1. We start with generalization of Banach - Alaoglu Theorem. Let $(E, \|\cdot\|)$ be a reflexive and separable Banach space. Prove that if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in E, there is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converging weakly in E.

2. Let $F : E \to \mathbb{R}$ be a convex function. We say that F is strongly lower semicontinuous if

$$F(x) \leq \liminf_{n\to\infty} F(x_n) \text{ whenever } x_n \to x.$$

   Similarly, we say that F is weakly lower semicontinuous if

$$F(x) \leq \liminf_{n\to\infty} F(x_n) \text{ whenever } x_n \rightharpoonup x.$$

   Use Riesz Lemma (Problem W7 in Problem Set 6) to deduce that if F is strongly lower semicontinuous and convex, then F is weakly lower semicontinuous.

3. Let $(E, \|\cdot\|)$ be a reflexive and separable Banach space. Let $A \subset E$ be a nonempty, closed, bounded and convex subset of E. Let $\varphi : A \to \mathbb{R}$ be a convex and strongly lower semicontinuous function. Prove that φ attains its minimum on A, i.e. there is some $x_0 \in A$ such that $\varphi(x_0) = \inf_{x\in A}\varphi(x)$.

4. Prove that if the subset $A \subset E$ in part (3) is not bounded, then the assertion is still valid under additional hypothesis on φ. Namely,

$$\lim_{x\in A, \|x\|\to\infty} \varphi(x) = \infty.$$

Remark 1. The techniques developed in this Problem form foundations for recent research in the theory of optimal transport (see *Optimal Transport for Applied Mathematicians* by F. Santambrogio) and theoretical materials science (see, for instance, *Calculus of variations* by F. Rindler).

Remark 2. Banach-Alaoglu Theorem holds in much more general situations and is usually formulated for the so-called weak$*$-convergence (this type of convergence coincides with the weak one in reflexive spaces).

7 Spectrum of a bounded operator is nonempty.

Announced: 20/12/2019, Deadline: 16/01/2020.

Let $A : H \to H$ be a bounded linear operator on the complex Hilbert space.

(a) Prove that $\sigma(A)$ is nonempty.
Hint: Try to combine Liouville Theorem on $\mathbb{C}$ with the inverse $(A - \lambda I)^{-1}$. Please, do not generalize the whole theory of analytic functions to the H-valued case.

(b) Show that the assertion does not hold for real Hilbert spaces, i.e. find operator $A : H \to H$ on real Hilbert space H such that $\sigma(A)$ is empty.

8 Fredholm alternative.

Announced: 20/12/2019, Deadline: 16/01/2020.

A fundamental result in the theory of compact operators is *Fredholm alternative*. Let $A : X \to X$ where A is a compact linear operator and X is a Banach space. Then, for $\lambda \neq 0$, Fredhold alternative asserts that precisely one of the following holds:

(A) There is a nontrivial $x \in X$ such that $Ax = \lambda x$.

(B) The operator $A - \lambda I$ has a bounded inverse.

This is some generalization of what one is usually taught at the Linear Algebra course: if $A \in \mathbb{R}^{n \times n}$ then $Ax = b$ has the unique solution if and only if $Ax = 0$ is satisfied by $x = 0$ only.

Warning: Do not apply spectral characterization of compact operators in parts (b), (c) and (d)!

Advice: It may be quite confusing to speak about eigenvalueas and spectrum for Banach space X as this stuff was introduced in the framework of Hilbert spaces. However, all definitions can be easily generalized to the case of Banach space.

(a) Find examples showing that both assumptions (A is compact, $\lambda \neq 0$) are necessary.

(b) Prove that if λ is not an eigenvalue of A, there is a constant $c > 0$ such that for all $x \in X$:

$$\|(A - \lambda I)x\| \geq c\|x\|.$$

(c) Prove that if λ is not an eigenvalue of A, then the image of $A - \lambda I$ is closed in X.

(d) Use Riesz Lemma to prove Fredholm alternative.

9 Marcinkiewicz Interpolation Theorem.

Announced: 1/01/2020, Deadline: 23/01/2020.

In the tutorials we have used Riesz–Thorin Interpolation Theorem to draw some conclusions about boundedness of convolution and Fourier transform. Another result in this spirit is Marcinkiewicz Interpolation Theorem which, briefly speaking, asserts that if $T : L^p \to L^p$ and $T : L^q \to L^q$ then $T : L^r \to L^r$ for all $p < r < q$. However, the result of Marcinkiewicz is valid in much more general setting and we will use this opportunity to introduce *weak L^p spaces*.

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. We write that $f \in L^{p,\infty}$ if the norm

$$\|f\|_{p,\infty} = \sup_{\lambda > 0} \lambda\, \mu(|f(x)| > \lambda)^{\frac{1}{p}} < \infty$$

for $p \in [1, \infty)$ and $\|f\|_{p,\infty} = \|f\|_\infty$. The space $L^{p,\infty}$ is sometimes called weak L^p space but we remark that it does not have anything to do with weak convergence in L^p. As some preliminary exercises, establish the following:

(A) If $f \in L^p$, then $f \in L^{p,\infty}$.

(B) There is a function $f \in L^{p,\infty}$ such that $f \notin L^p$ (case $X = \mathbb{R}$ is sufficient).

(C) If $f \in L^p$ and $1 \le p_0 < p < p_1 \le \infty$ one can find $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$ such that $f = f_0 + f_1$.

(D) If $\varphi : [0, \infty) \to \mathbb{R}$ is an increasing and differentiable function with $\varphi(0) = 0$ then

$$\int_X \varphi(|f|(x))\, d\mu(x) = \int_0^\infty \varphi'(t)\mu(|f| > t)\, dt.$$

In particular, consider $\varphi(t) = t^p$.

Now, we state the Marcinkiewicz Interpolation Theorem.

Theorem. Let T be an operator from $L^p + L^q$ to a complex-valued measurable functions where $1 \le p, q \le \infty$. Assume that T is sublinear i.e. there is a constant $c > 0$ such that

$$|T(f+g)| \le c|Tf| + c|Tg|, \qquad |T(\lambda f)| = |\lambda|\,|Tf| \text{ for all } \lambda \in \mathbb{C}.$$

Moreover, assume that T is a bounded operator as a map from L^p to $L^{p,\infty}$ and from L^q to $L^{q,\infty}$. Then, T is a bounded operator from L^r to L^r for all $r \in (p, q)$.

(E) Use (A), (C) and (D) to prove Marcinkiewicz Interpolation Theorem.

Remark: One of the numerous applications of this result is L^p boundedness of maximal operator for $1 < p < \infty$. More precisely, for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ one can define

$$Mf(x) = \sup_B \frac{1}{|B|} \int_B |f(x)|\, dx$$

where the supremum is taken over all balls containing x. It is trivial that $M : L^\infty \to L^\infty$ and it is also not so difficult to see that $M : L^1 \to L^{1,\infty}$ and so Marcinkiewicz Theorem implies that $M : L^p \to L^p$ for $1 < p < \infty$. The important point here is that it not true that $M : L^1 \to L^1$. Maximal operators are useful in demonstrating pointwise properties of L^p functions like Lebesgue Differentiation Theorem.

10 Fourier series does not converge for every continuous function.

Announced: 1/01/2020, Deadline: 23/01/2020.

Let $S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i kx}$ be the partial sum of the Fourier series of function f defined on $[0, 1)$ and extended periodically to $\mathbb{R}$. Recall that:

$$\hat{f}(k) = \int_0^1 f(t) e^{-2\pi i kt}\, dt.$$

In the tutorials, we have seen that if f is continuous and satisfies Dini's condition:

$$\text{there is } \delta > 0 \text{ such that } \int_{|t|<\delta} \frac{|f(t+x) - f(x)|}{|t|}\, dt < \infty$$

then $S_N f(x) \to f(x)$ as $N \to \infty$. Here, we will see that continuity of f is not sufficient for pointwise convergence of $S_N f(x)$ to $f(x)$.

(A) Let $D_N(x) = \sum_{k=-N}^{N} e^{2\pi i kx}$ be the Dirichlet kernel. Prove that when N is large, $\|D_N\|_1$ behaves like $\log N$. Hence, as $N \to \infty$, $\|D_N\|_1 \to \infty$.

(B) Prove that there is a continuous 1-periodic continuous function $f \in C(\mathbb{R})$ such that $|S_N f(0)|$ diverges to ∞.