Comments on Big Homework 4

Problem 2 This is important problem that may come back in a specified setting (particular choice of f etc...)

(a) This is trivial. [convex, closed subset]

(b) Here, (a) was actually a hint. We shall apply Hahn-Banach with epigraph of f in $E\times\mathbb{R}$.

First, note that if $f\in(E\times\mathbb{R})^*$ then $f((x,y)) = f((x,0)+(0,y)) =$

$= f((x,0)) + f(0,y) = g(x) + y f(0,1) = g(x) + y\cdot c$.

Fix $z\in E$. Clearly, the point $(z, \varphi(z)-\frac{1}{n}) \notin \operatorname{epi}\varphi$. By H-B, there is $f_{n,z}\in(E\times\mathbb{R})^*$, $f_{n,z}(x,y) = g_n(x) + y\cdot c_n$ such that

$$(*)\quad g_n(w) + \varphi(w)\cdot c_n > \lambda > g_n(z) + \left(\varphi(z)-\frac{1}{n}\right)c_n \qquad \forall\, w\in E$$

First, we take $w=z$ to get

$$\varphi(z)c_n > \left(\varphi(z)-\frac{1}{n}\right)c_n \implies \boxed{c_n>0}$$

Then, we use $(*)$ once again to get

$$(**)\quad \varphi(w) > \frac{1}{c_n} g_n(z-w) + \varphi(z) - \frac{1}{n}$$

We let $g_{n,z}(w) = \frac{1}{c_n} g_n(z-w) + \varphi(z) - \frac{1}{n}$. From $(**)$ we have $\varphi(w) > g_{n,z}(w)$ and $\varphi(z) > g_{n,z}(z) = \varphi(z) - \frac{1}{n}$.

This shows that $\varphi = \sup_{n,z} g_{n,z}$. $\square$

Problem 4

We start, as always, with the point spectrum.

- point spectrum for R: $Rx = \lambda x \Rightarrow (0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$ $\Rightarrow x = 0$ as R has empty point spectrum.

- point spectrum for L: $Lx = \lambda x \Rightarrow (x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \dots)$ so $x_2 = \lambda x_1$, $x_3 = \lambda x_2 = \lambda^2 x_1$ and in general $x_n = \lambda^n x_1$. Sequence defined in that way is in ℓ^2 iff $|\lambda| < 1$. Hence, $\sigma_p(L) = \{|\lambda| < 1\}$

Now, we note that $\|R\| = \|L\| = 1$ so $\sigma(L), \sigma(R) \subset \{|\lambda| \leq 1\}$.
We also note that $R^* = L$ so $(R - \lambda I)^* = L - \bar{\lambda} I$ and we deduce
$\ker((R-\lambda I))^\perp = \overline{\mathrm{im}((R-\lambda I)^*)} = \overline{\mathrm{im}(L - \bar{\lambda} I)}$

As we already know $\ker(R - \lambda I) = \{0\}$ so $\overline{\mathrm{im}(L - \bar{\lambda} I)} = H$ but we also know that $|\lambda| = 1$ has to be in $\sigma(L)$ so we conclude that

as $\sigma(L)$ is closed and $\{|\lambda|<1\} \subset \sigma(L)$

$$\sigma_p(L) = \{|\lambda| < 1\} \qquad \sigma_r(L) = \emptyset.$$
$$\sigma_c(L) = \{|\lambda| = 1\}$$

Now, we study operator R. Again, we have $L^* = R$. so we have $\ker(L - \lambda I) = \mathrm{im}(R - \bar{\lambda} I)^\perp$. We know that $\ker(L - \lambda I) \neq \{0\}$ for $|\lambda| < 1$ so $\overline{\mathrm{im}(R - \bar{\lambda} I)} \neq H$. This shows that $\{|\lambda| < 1\} \subset \sigma(R)$ and $\{|\lambda| < 1\} \subset \sigma_p(R) \cup \sigma_r(R)$ (it cannot be in the continuous part as $\overline{\mathrm{im}(R - \lambda I)} \neq H$. But R has empty point spectrum so $\sigma_p(R) \supset \{|\lambda| < 1\}$.

Again, as spectrum is closed $\sigma(R) = \{|\lambda| \leq 1\}$ and we are left to decide to which part belongs $\{|\lambda| = 1\}$. If $|\lambda| = 1$ and $\overline{\mathrm{im}(R - \lambda I)} \neq H$, then $\ker(L - \lambda I)^{\perp} \neq H$ so

$\ker(L - \lambda I) \neq \{0\}$ and $\lambda \in \sigma_p(L)$ but $\sigma_p(L) = \{|\lambda| < 1\} \Rightarrow$ Contradiction. Hence $\{|\lambda| = 1\} = \sigma_c(R)$. The conclusion is

$$\sigma_p(R) = \emptyset$$

$$\sigma_r(R) = \{|\lambda| < 1\}$$

$$\sigma_c(R) = \{|\lambda| = 1\}.$$