Functional Analysis (WS 19/20), Big Homework 1

deadline: 31/10/2019 (group no. 1), 5/11/2019 (group no. 2)

Important: Each problem should be solved on a separate piece of paper signed with your name, student id number and group number (1 or 2).

1. Consider set $C^1[0,1]$ of continuously differentiable functions on $[0,1]$. We define

$$\|f\|_C := |f(0)|^{1} + \sup_{x\in[0,1]} |f'(x)|$$

and

$$\|f\|_D := \left(\int_0^1 (f(x))^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 (f'(x))^2 dx\right)^{\frac{1}{2}}.$$

Are $(C^1[0,1], \|f\|_C)$ and $(C^1[0,1], \|f\|_D)$ normed spaces? Are they Banach spaces?

2. Let $1 \le p \le \infty$ and $T : l^p \to l^p$ be defined with

$$T\left((a_n)_{n\ge1}\right) = (a_{n+1} - a_n)_{n\ge1}.$$

Check that T is well - defined (i.e. $T\left((a_n)_{n\ge1}\right) \in l^p$ whenever $(a_n)_{n\ge1} \in l^p$), prove that it is a bounded linear operator and compute its norm.

3. Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Suppose that D is a dense linear subspace of X and $T : (D, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a bounded linear operator. Prove that T has a unique bounded[2] extension to X which preserves the norm. *Hint:* If $x \in X \setminus D$, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$.

   By an extension of T to X which preserves the norm, we mean an operator $\tilde{T} : X \to Y$ such that $\tilde{T} = T$ on $D \subset X$ and $\|T\|_{\mathcal{L}(D,Y)} = \|\tilde{T}\|_{\mathcal{L}(X,Y)}$.[3]

4. Let $(x_n)_{n\ge1}$ be a sequence of real numbers such that whenever $(y_n)_{n\ge1}$ is a real sequence converging to 0 we have that $\sum_{n\ge1} x_n y_n$ is convergent. Prove that $\sum_{n\ge1} |x_n|$ is convergent. *Hint*: for $y \in c_0$, consider $T_n \in (c_0)^*$ defined with $T_n(y) = \sum_{k=1}^{n} x_k y_k$.

[1] Update on 23.10.2019: $f(0)$ replaced with $|f(0)|$.

[2] Update on 26.10.2019: I added information that extension is bounded (but it actually follows from condition $\|T\|_{\mathcal{L}(D,Y)} = \|\tilde{T}\|_{\mathcal{L}(X,Y)}$.

[3] Update on 19.10.2019: I added clarification what we mean by extension of operator.